1. (16 points total) Consider the differential equation
\[
\frac{dy}{dx} = 2y^{1/2}
\]
(a) Find the solution that passes through \((0, 4)\).
\[
\int \frac{dy}{\sqrt{y}} = \int dx
\]
\[
y^{1/2} = x + C
\]
\[
2 = 0 + C \quad \Rightarrow \quad C = 2
\]
\[
y = (x + 2)^2
\]
Note: \((x - 2)^2\) is not a solution.

(b) The function \(y(x) = 0\) is a solution of this differential equation. Show that your solution in (a) intersects this solution and explain why this does not contradict the existence-uniqueness theorem.

Solution from (a) hits 0 at \(x = -2\). So the two solutions intersect at \((-2, 0)\). But 2 \(y^{1/2}\) is not defined for \(y < 0\), so theorem does not apply to the initial condition \((-2, 0)\).
2. (20 points) Consider the differential equation \( \frac{dy}{dx} = y(2 - y) \).

(a) Find \( a \) so that the following is a solution.

\[
y(x) = \frac{2}{1 + e^{-ax}}
\]

\[
y'(x) = \frac{2 a e^{-ax}}{(1 + e^{-ax})^2}
\]

\[
y'(2 - y) = \frac{2}{1 + e^{-ax}} \left( 2 - \frac{2}{1 + e^{-ax}} \right)
= \frac{2 \left( 2 + 2 e^{-ax} - 2 \right)}{(1 + e^{-ax})^2}
= \frac{4 e^{-ax}}{(1 + e^{-ax})^2}
\]

So \( 2a = 2 \), i.e., \( a = 1 \).

(b) Which of the following is a symmetry of the dif. eq.

(i) horizontal translation, i.e., if \( y(x) \) is a solution then \( y(x - c) \) is too.

\( \text{YES} \)

(ii) vertical translation, i.e., if \( y(x) \) is a solution then \( y(x) + c \) is too.

\( \text{NO} \)

(c) Find the solution through the point \((1, 1)\) Hint: Note that the solution in part (a) passes through \((0, 1)\).

Let \( y'(x) = \frac{2}{1 + e^{-2x}} \).

It is a solution and \( y(0) = 1 \).

By (b), \( \tilde{y}(x) = y(x - c) \) is also a solution.

Choose \( c \) so \( \tilde{y}(1) = 1 \). \( \tilde{y}(0) = 1 \), so \( \tilde{y}(1) = y(1 - c) \). So \( 2 \)

\[
y(x) = \frac{2}{1 + e^{-2(x-1)}}
\]
3. (18 points) Consider the differential equation,

\[
\frac{dy}{dt} = y(2 - e^{y/b})
\]

The parameter \( b \) can be any nonzero number. Find the equilibrium solutions and determine if they are stable, unstable or semistable.

\[
y'(2 - e^{y/b}) = 0
\]

\[
y = 0 \quad \text{or} \quad e^{y/b} = 2
\]

\[
\text{I.C.} \quad y = b \ln 2
\]

\[
y' = 2 - e^{y/b} - y e^{y/b} + \frac{1}{b}
\]

\[
y'(0) = 2 - 1 - 0 = 1
\]

So, \( y = 0 \) is always \underline{unstable}

\[
y'(b \ln 2) = 2 - e^{b \ln 2} - b \ln 2 e^{b \ln 2} + \frac{1}{b}
\]

\[
= -2 b \ln 2 < 0
\]

So, \( b \ln 2 \) is always \underline{stable}
4. (16 points) For the differential equation
\[ \frac{dy}{dx} = e^{x+y} - 1 \]

(a) Where are the solution curves increasing and where are they decreasing?
\[ \frac{dy}{dx} > 0 \implies e^{x+y} > 1 \iff x+y < 0 \]
\[ \frac{dy}{dx} < 0 \implies e^{x+y} < 1 \iff x+y > 0 \]

(b) Where are the solution curves concave up and where are they concave down?
\[ \frac{d^2y}{dx^2} = e^{x+y} \left(1 + \frac{dy}{dx}\right) = e^{x+y} \left(1 + e^{x+y} - 1\right) \]
\[ = e^{2(x+y)} > 0 \quad \text{always} \]
So \underline{always concave up}

5. (12 points) For this question, you need not show any work. No partial credit will be given on this one. Let \( y(x) \) be a solution of the differential equation
\[ \frac{dy}{dx} = \frac{1}{(y-x)^2} + 1 \]

(a) Let \( \bar{y}(x) = y(-x) \). Is \( \bar{y} \) a solution?
\[ \text{NO} \]

(b) Let \( \bar{y}(x) = -y(x) \). Is \( \bar{y} \) a solution?
\[ \text{NO} \]

(c) Let \( \bar{y}(x) = -y(-x) \). Is \( \bar{y} \) a solution?
\[ \text{YES} \]
6. (15 points) In the homework you solved the differential equation 
\( y' = \frac{1}{2} (1 - y^2) \) with the initial condition \( y(0) = 0 \) and found, hopefully,

\[
y(x) = \frac{e^x - 1}{e^x + 1}
\]

Now consider the differential equation \( f' = 4 - f^2 \) with the initial condition \( f(0) = 0 \). Find the solution. Hint: There are two ways to do this. You can ignore the above formula for \( y(x) \) and just solve the dif eq for \( f \). Of you can use scaling.

Try 
\[
\begin{align*}
f(a) &= a \cdot y(bx) \\
\frac{df}{dx} &= a b \cdot y'(bx) \\
&= a b \cdot \frac{1}{2} \left( 1 - y^2(bx) \right) \\
&= \frac{a b}{2} \left( 1 - \left( \frac{f(x)}{a} \right)^2 \right) \\
&= \frac{a b}{2} - \frac{b}{2a} \cdot f(x)
\end{align*}
\]

want 
\[
4 - f^2
\]

So 
\[
\frac{a b}{2} = 4, \quad \frac{b}{2a} = 1
\]

\( \Rightarrow \) 
\[
a = 2, \quad b = 4
\]

So 
\[
\begin{align*}
f(\pi) &= 2 \cdot y(4 \pi) = 2 \cdot \frac{e^{4 \pi} - 1}{e^{4 \pi} + 1}
\end{align*}
\]