

Math 250A (Kennedy) - Exam 2 - Fall '07

SHOW YOUR WORK. Correct answers with no work will get no credit.

1. (16 points total) Consider the differential equation

$$\frac{dy}{dx} = 2y^{1/2}$$

(a) Find the solution that passes through (0, 4).

$$\int \frac{dy}{2y^{1/2}} = \int dx$$

$$y^{1/2} = x + C$$

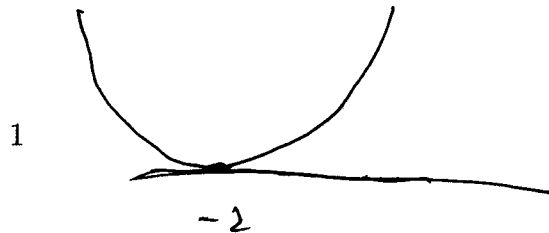
$$2 = 0 + C \Rightarrow C = 2$$

$$\boxed{y = (x+2)^2}$$

Note: $(x-2)^2$ is not a solution

(b) The function $y(x) = 0$ is a solution of this differential equation. Show that your solution in (a) intersects this solution and explain why this does not contradict the existence-uniqueness theorem.

Solution from (a) has 0 at $x = -2$.
 So the two solutions intersect at $(-2, 0)$. But $2y^{1/2}$ is not defined for $y < 0$, so theorem does not apply to the initial condition $(-2, 0)$.



2. (20 points) Consider the differential equation $\frac{dy}{dx} = y(2-y)$.
 (a) Find a so that the following is a solution.

$$y(x) = \frac{2}{1+e^{-ax}}$$

$$y'(x) = \frac{2a e^{-ax}}{(1+e^{-ax})^2}$$

$$\begin{aligned} y(2-y) &= \frac{2}{1+e^{-ax}} \left(2 - \frac{2}{1+e^{-ax}} \right) \\ &= \frac{2(2 + 2e^{-ax} - 2)}{(1+e^{-ax})^2} = \frac{4e^{-ax}}{(1+e^{-ax})^2} \end{aligned}$$

So need $2a = 4$, i.e., $\boxed{a = 2}$

- (b) Which of the following is a symmetry of the dif. eq.
 (i) horizontal translation, i.e., if $y(x)$ is a solution then $y(x-c)$ is too.

YES

- (ii) vertical translation, i.e., if $y(x)$ is a solution then $y(x)+c$ is too.

NO

- (c) Find the solution through the point $(1, 1)$ Hint: Note that the solution in part (a) passes through $(0, 1)$.

$$\text{Let } y(x) = \frac{2}{1+e^{-2x}}$$

It is a solution and $y(0) = 1$.

By (b), $\bar{y}(x) = y(x-c)$ is also a solution.

Choose c so $\bar{y}(1) = 1$. $y(0) = 1$, so

$$\bar{y}(1) = y(1-c). \text{ So } 2 \text{ let } c = 1.$$

$$\bar{y}(x) = y(x-1) = \boxed{\frac{2}{1+e^{-2(x-1)}}$$

3. (18 points) Consider the differential equation,

$$\frac{dy}{dt} = y(2 - e^{y/b})$$

The parameter b can be any nonzero number. Find the equilibrium solutions and determine if they are stable, unstable or semistable.

$$y(2 - e^{y/b}) = 0$$

\Rightarrow $\boxed{y = 0}$ or $e^{y/b} = 2$
i.e., $\boxed{y = b \ln 2}$

$$g' = 2 - e^{y/b} - y e^{y/b} \frac{1}{b}$$

$$g'(0) = 2 - 1 - 0 = 1$$

So 0 is always unstable

$$g'(b \ln 2) = 2 - e^{\ln 2} - b \ln 2 e^{\ln 2} \frac{1}{b}$$
$$= -2 \ln 2 < 0$$

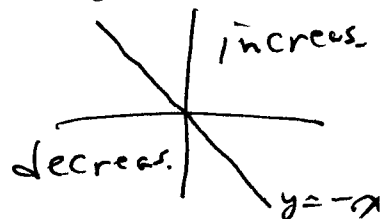
So $b \ln 2$ is always stable

4. (16 points) For the differential equation

$$\frac{dy}{dx} = e^{x+y} - 1$$

(a) Where are the solution curves increasing and where are they decreasing?

$$\begin{aligned} \frac{dy}{dx} > 0 & \quad \text{if} \quad e^{x+y} > 1 \quad \Leftrightarrow \quad x+y > 0 \\ \frac{dy}{dx} < 0 & \quad \text{if} \quad e^{x+y} < 1 \quad \Leftrightarrow \quad x+y < 0 \end{aligned}$$



(b) Where are the solution curves concave up and where are they concave down?

$$\begin{aligned} \frac{d^2y}{dx^2} &= e^{x+y} \left(1 + \frac{dy}{dx} \right) = e^{x+y} (1 + e^{x+y} - 1) \\ &= e^{2(x+y)} > 0 \quad \text{always} \\ \text{So} \quad &\underline{\text{always concave up}} \end{aligned}$$

5. (12 points) For this question, you need not show any work. No partial credit will be given on this one. Let $y(x)$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{(y-x)^2} + 1$$

(a) Let $\bar{y}(x) = y(-x)$. Is \bar{y} a solution?

No

(b) Let $\bar{y}(x) = -y(x)$. Is \bar{y} a solution?

No

(c) Let $\bar{y}(x) = -y(-x)$. Is \bar{y} a solution?

2 YES

6. (15 points) In the homework you solved the differential equation $y' = \frac{1}{2}(1 - y^2)$ with the initial condition $y(0) = 0$ and found, hopefully,

$$y(x) = \frac{e^x - 1}{e^x + 1}$$

Now consider the differential equation $f' = 4 - f^2$ with the initial condition $f(0) = 0$. Find the solution. Hint: There are two ways to do this. You can ignore the above formula for $y(x)$ and just solve the dif eq for f . Or you can use scaling.

Try $f(x) = a y(bx)$

$$f'(x) = ab y'(bx)$$

$$= ab \frac{1}{2} (1 - y^2(bx))$$

$$= \frac{ab}{2} \left(1 - \left[\frac{f(x)}{a} \right]^2 \right)$$

$$= \frac{ab}{2} - \frac{b}{2a} f^2(x)$$

Want $4 - f^2$

So $\frac{ab}{2} = 4, \quad \frac{b}{2a} = 1$

$\Rightarrow a = 2, \quad b = 4$

So $f(x) = 2 y(4x) = 2 \frac{e^{4x} - 1}{e^{4x} + 1}$