Math 250b (Kennedy) - Quiz 1 Solutions - Spring '08

1. Calculate the following integral if it converges

$$\int_0^\infty \frac{e^{-x}}{1+e^{-x}}\,dx$$

Solution: Let $u = 1 + e^{-x}$. Then $du = -e^{-x}dx$. So

$$\int \frac{e^{-x}}{1+e^{-x}}\,dx = \int \frac{-1}{u}\,du = -\ln|u| = -\ln(1+e^{-x})$$

So

$$\int_0^\infty \frac{e^{-x}}{1+e^{-x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^{-x}}{1+e^{-x}}\,dx$$
$$= \lim_{b\to\infty}[-\ln(1+e^{-b}) + \ln(1+e^{-0})] = -\lim_{b\to\infty}\ln(1+e^{-b}) + \ln(2)$$

As $b \to \infty$, $e^{-b} \to 0$, so $\ln(1+e^{-b}) \to \ln(1) = 0$.

So the original integral converges to $\ln(2)$.

2. For each of the following two improper integrals determine if it converges or diverges. You need **not** justify your answer.

$$\int_1^\infty \frac{x}{x^2+10}\,dx$$

Solution: For large x

$$\frac{x}{x^2+10} \approx \frac{x}{x^2} = \frac{1}{x}$$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, the original integral also **diverges**.

$$\int_0^2 \frac{2-\cos(x)}{\sqrt{x}}\,dx$$

Solution: $2-\cos(x)$ oscillates between 1 and 3. In particular it is always at most 3. So

$$\frac{2-\cos(x)}{\sqrt{x}} \leq \frac{3}{\sqrt{x}}$$

Since $\int_0^2 1/\sqrt{x}\,dx$ coverges, the original integral also **converges**.