Math 250b (Kennedy) - Quiz 8 - Spring '08

Each of the following systems has only one equilibrium. For each system,
(a) Find the equilibrium
(b) Classify it as stable node, unstable node, saddle, stable spiral, unstable spiral, periodic (center).
(c) Find all the straight line orbits. (Some systems might not have any.)

You can use the phase plane program to check your answer, but for full credit you must derive your answer analytically.

1. $x' = 4x - 10y, \quad y' = 2x - 4y.$

(a) Clearly $(0, 0)$ is an equilibrium and we were given that there is only one.
(b) To find the roots:

$$\begin{aligned} 0 &= det \begin{pmatrix} 4-r & -10 \\ 2 & -4-r \end{pmatrix} = (4-r)(-4-r) + 20 \\ &= r^2 + 4 \end{aligned}$$

So $r = \pm 2i$. Thus it is perioidic.
(c) Since its perioidic, there are no straight line orbits.

2. $x' = -x - y + 1, \quad y' = x - y - 3.$

(a) To find the equilibrium we have

$$\begin{aligned} 0 &= -x - y + 1 \\ 0 &= x - y - 3 \end{aligned}$$

Adding these equations gives $y = -1$. Back substituting we get $x = 2$. So the equilibrium is $(2, -1)$.
(b)

$$\begin{aligned} 0 &= det \begin{pmatrix} -1-r & -1 \\ 1 & -1-r \end{pmatrix} = (-1-r)(-1-r) + 1 = \\ &= r^2 + 2r + 2 \end{aligned}$$

This has roots $-1 \pm i$. So we have a stable spiral.
(c) Since its a stable spiral, there are no straight line orbits.

3. $x' = 2x + 4y, \quad y' = x - y.$

(a) Clearly $(0, 0)$ is an equilibrium and we were given that there is only one.

(b) To find the roots:

$$\begin{aligned} 0 &= det \begin{pmatrix} 2-r & 4 \\ 1 & -1-r \end{pmatrix} = (2-r)(-1-r) - 4 \\ &= r^2 - r - 6 \end{aligned}$$

which has roots $r = -2, 3$. Thus it is a saddle.
(c) (eigenvector approach) The eigenvector for $\lambda = -2$ satisfies

$$\begin{aligned} 2x + 4y &= -2x \\ x - y &= -2y \end{aligned}$$

As always the equations are degenerate and can be written as $4y = x$. So the line is $y = x/4$.

The eigenvector for $\lambda = 3$ satisfies

$$\begin{aligned} 2x + 4y &= 3x \\ x - y &= 3y \end{aligned}$$

As always the equations are degenerate and can be written as $y = -x$. So the line is $y = -x$.
(c) (other approach) The differential eq for trajectories is

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{x-y}{2x+4y}$$

Assume $y = mx$. Then this becomes

$$m = \frac{x - mx}{2x + 4mx} = \frac{1-m}{2+4m}$$

This is $m(2 + 4m) = 1 - m$, i.e., $4m^2 + 3m - 1$, which has roots $m = -1$, and $m = 1/4$. So we get the same lines $y = x/4$ and $y = -x$.