Math 250b (Kennedy) - Quiz 9 Solutions - Spring ’08

Variation of parameters: $z'_1x_1 + z'_2x_2 = 0, \quad z'_1x'_1 + z'_2x'_2 = \frac{f(t)}{a_2}$

1. (a) $t^2$ is one solution of the homogeneous equation $t^2x'' - 2x = 0$. Find another (linearly independent) solution of the homogeneous equation.

**First method:** Try $x = t^p$. Then $x' = pt^{p-1}$ and $x'' = p(p-1)t^{p-2}$. So the dif. eq. becomes

$$t^2p(p-1)t^{p-2} - 2t^p = 0$$

which is true if $p(p-1) - 2 = 0$. This has solutions $p = 2$ and $p = -1$. So the other solution is $t^{-1}$.

**Second method:** Use reduction of order. Take $x = t^2z$. Then

$$x' = 2tz + t^2z'$$
$$x'' = 2z + 4tz' + t^2z''$$

So the dif. eq. becomes

$$t^2(2z + 4tz' + t^2z'') - 2t^2z = 0$$

which simplifies to

$$z'' + \frac{4}{t}z' = 0$$

You can solve this by an integrating factor (its $t^4$) or separation of variables.

Letting $u = z'$, separation of variables gives

$$\frac{du}{dt} = -\frac{4u}{t}$$
$$\frac{du}{u} = -\frac{4dt}{t}$$
$$\int \frac{du}{u} = \int -\frac{4dt}{t}$$
$$\ln |u| = -4\ln |t| + C$$
$$u = e^{C/t^4}$$

We only need one solution, so take $C = 1$. So $z' = t^{-4}$, so $z = -\frac{1}{3}t^{-3}$, and $x = -\frac{1}{3}t^{-1}$. So $t^{-1}$ is another solution.
(b) Use the two solutions of the homogeneous eq. from (a) to find the general solution of $t^2x'' - 2x = 3t^2$.

**First method:** Use reduction of order. $x = t^2z$. This starts the same as in part (a), but now the dif. eq. becomes

$$t^4z'' + 4t^3z' = 3t^2$$

$$z'' + \frac{4}{t}z' = 3t^{-2}$$

The integrating factor is the same as in part (a),

$$\exp(\int \frac{4}{t} dt) = \exp(4 \ln t) = t^4$$

Multiply the equation by this factor:

$$t^4z'' + 4t^3z' = 3t^2$$

$$(t^4z')' = 3t^2$$

Integrating, $t^4z' = t^3$. So $z' = 1/t$. So $z = \ln(t)$. So $x = t^2\ln(t)$ is a solution of the inhomogeneous equation. So the general solution is $x = t^2\ln(t) + c_1 t^2 + c_2 t^{-1}$.

**Second method:** Use variation of parameters.

$$t^2z_1' + t^{-1}z_2' = 0$$

$$2tz_1' - t^{-2}z_2' = 3$$

Solving for $z_1'$ we find $z_1' = 1/t$. So $z_1 = \ln(t) + C_1$. Solving for $z_2'$ we find $z_2' = -t^2$. So $z_2 = -\frac{1}{3}t^3 + C_2$. Thus we get

$$x = t^2(\ln(t) + C_1) + t^{-1}(\frac{1}{3}t^3 + C_2)$$

$$= t^2 \ln(t) + C_1 t^2 - \frac{1}{3}t^2 + C_2 t^{-1}$$

This agrees with the previous method with $c_2 = C_2$ and $c_1 = C_1 - 1/3$. 