Math 250b (Kennedy) - Quiz 9 Solutions - Spring '08

Variation of parameters: $z_1'x_1 + z_2'x_2 = 0,$ $z_1'x_1' + z_2'x_2' = \frac{f(t)}{a_2}$

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1. (a) t^2 is one solution of the homogeneous equation $t^2x'' - 2x = 0$. Find another (linearly independent) solution of the homogeneous equation.

First method: Try $x = t^p$. Then $x' = pt^{p-1}$ and $x'' = p(p-1)t^{p-2}$. So the dif. eq. becomes

$$t^2p(p-1)t^{p-2} - 2t^p = 0$$

which is true if $p(p-1) - 2 = 0$. This has solutions $p = 2$ and $p = -1$. So the other solution is t^{-1}.

Second method: Use reduction of order. Take $x = t^2z$. Then

$$\begin{aligned} x' &= 2tz + t^2z' \\ x'' &= 2z + 4tz' + t^2z'' \end{aligned}$$

So the dif. eq. becomes

$$t^2(2z + 4tz' + t^2z'') - 2t^2z = 0$$

which simplifies to

$$z'' + \frac{4}{t}z' = 0$$

You can solve this by an integrating factor (its t^4) or separation of variables. Letting $u = z'$, separation of variables gives

$$\begin{aligned} \frac{du}{dt} &= \frac{-4u}{t} \\ \frac{du}{u} &= \frac{-4dt}{t} \\ \int \frac{du}{u} &= \int \frac{-4dt}{t} \\ \ln|u| &= -4\ln|t| + C \\ u &= e^Ct^{-4} \end{aligned}$$

We only need one solution, so take $C = 1$. So $z' = t^{-4}$, so $z = \frac{-1}{3}t^{-3}$, and $x = \frac{-1}{3}t^{-1}$. So t^{-1} is another solution.

(b) Use the two solutions of the homogeneous eq. from (a) to find the general solution of $t^2x'' - 2x = 3t^2$.

First method: Use reduction of order. $x = t^2 z$. This starts the same as in part (a), but now the dif. eq. becomes

$$\begin{aligned} t^4 z'' + 4t^3 z' &= 3t^2 \\ z'' + \frac{4}{t} z' &= 3t^{-2} \end{aligned}$$

The integrating factor is the same as in part (a),

$$\exp(\int \frac{4}{t} dt) = \exp(4 \ln t) = t^4$$

Multiply the equation by this factor:

$$\begin{aligned} t^4 z'' + 4t^3 z' &= 3t^2 \\ (t^4 z')' &= 3t^2 \end{aligned}$$

Integrating, $t^4 z' = t^3$. So $z' = 1/t$. So $z = \ln(t)$. So $x = t^2 \ln(t)$ is a solution of the inhomogeneous equation. So the general solution is $x = t^2 \ln(t) + c_1 t^2 + c_2 t^{-1}$.

Second method: Use variation of parameters.

$$\begin{aligned} t^2 z_1' + t^{-1} z_2' &= 0 \\ 2t z_1' - t^{-2} z_2' &= 3 \end{aligned}$$

Solving for z_1' we find $z_1' = 1/t$. So $z_1 = \ln(t) + C_1$. Solving for z_2' we find $z_2' = -t^2$. So $z_2 = -\frac{1}{3}t^3 + C_2$. Thus we get

$$\begin{aligned} x &= t^2(\ln(t) + C_1) + t^{-1}(-\frac{1}{3}t^3 + C_2) \\ &= t^2 \ln(t) + C_1\, t^2 - \frac{1}{3}t^2 + C_2\, t^{-1} \end{aligned}$$

This agrees with the previous method with $c_2 = C_2$ and $c_1 = C_1 - 1/3$.