Math 250B (Kennedy) - Exam 1 Solutions - Spring '08

SHOW YOUR WORK. Correct answers with no work will get no credit. There are 6 problems for a total of 100 points.

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

The error in the nth order Taylor polynomial has an absolute value of at most

$$\frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}$$

where M_{n+1} is the maximum of $|f^{(n+1)}|$ over the interval from a to x.

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1. (12 points) Determine if the following improper integral converges or diverges. Explain your reasoning.

$$\int_2^\infty \frac{dx}{\sqrt{x^3+1}}$$

SOLUTION:

We have

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$$

for all $x \geq 2$. The integral

$$\int_2^\infty \frac{dx}{x^{3/2}}$$

converges since $3/2 > 1$. So by the comparison test the original integral converges.

2. (24 points total) For each of the following series determine whether the series converges or diverges. Explain your reasoning, including giving the name of the test that you use.

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)^2}$$

FIRST SOLUTION: We have $1/(n+5)^2 \leq 1/n^2$ and we know $\sum_n 1/n^2$ converges, so by the *comparison test* the original series converges.

SECOND SOLUTION: We can use the *integral test*. $1/(x+5)^2$ is a positive, decreasing function and you can compute $\int_2^\infty 1/(x+5)^2$ to show that it converges. So the original series converges.

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

FIRST SOLUTION: As $n \to 0$, $n!/10^n$ does not conveges to 0. In fact it goes to ∞. So by the *stupid test* the series does not converge.

SECOND SOLUTION: Use the *ratio test*.

$$\frac{(n+1)!}{10^{n+1}} \frac{10^n}{n!} = \frac{n+1}{10} \tag{1}$$

The converges to ∞, which is not less than 1. So the series diverges.

$$\sum_{n=2}^{\infty} \frac{3^n}{2^{2n}}$$

FIRST SOLUTION: Since $3^n/2^{2n} = (3/4)^n$, it is a *geometric series* with $r = 3/4 < 1$. So it converges.

SECOND SOLUTION: Use the *ratio test*.

$$\frac{3^{n+1}}{2^{2(n+1)}} \frac{2^{2n}}{3^n} = \frac{3}{4}$$

Since this is less than 1, the series converges.

3. (24 points total)
(a) Find the third order Taylor polynomial about the origin of $\cos(x)\, e^x$.

SOLUTION: To third order,

$$\begin{aligned}\cos(x) &\approx 1 - \frac{x^2}{2} \\ e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}\end{aligned}$$

So

$$\cos(x)e^x \approx (1 - \frac{x^2}{2})(1 + x + \frac{x^2}{2} + \frac{x^3}{6}) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^2}{2} - \frac{x^3}{2} + \cdots$$

where $\cdots$ represents terms of order higher than three. We drop these terms and get

$$\cos(x)e^x \approx 1 + x - \frac{x^3}{3}$$

(b) Find the Taylor series of $x\exp(x^2)$ about the origin.

SOLUTION:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Substituting x^2 for x we have

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Mutiply this by x and we get

$$xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

4. (16 points total) Find the interval of convergence of the following series. (You don't have to determine what happens at the endpoints.)

$$\sum_{n=1}^{\infty} \frac{2^n (x+1)^n}{n+1}$$

SOLUTION: Use the ratio test

$$\frac{2^{n+1}|x+1|^{n+1}}{n+2} \frac{n+1}{2^n |x+1|^n} = 2|x+1| \frac{n+1}{n+2}$$

This converges to $2|x+1|$. So the series converges if $2|x+1| < 1$, i.e., $|x+1| < 1/2$. This is the interval $(-3/2, -1/2)$.

5. (12 points total) Suppose we approximate $e^{0.2}$ by

$$e^{0.2} \approx 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!}$$

Find a bound on the error in the approximation.
SOLUTION: This approximation is the fourth order Taylor polynomial about $a = 0$. To compute M_5 we need the fifth derivative of e^x which is just e^x. So

$$M_5 = \max_{0 \le x \le 0.2} e^x = e^{0.2} \tag{2}$$

So the bound on the error is

$$\frac{e^{0.2}(0.2)^5}{5!}$$

6. (12 points total) The force of gravity on an object of mass m at a height h above the surface of the earth is

$$F(h) = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth. If h is small compared to R, we can use Taylor series to write this as an expansion in powers of h/R :

$$F(h) = mg(c_0 + c_1\frac{h}{R} + c_2\frac{h^2}{R^2} + \cdots)$$

Find the coeffecients c_0, c_1 and c_2.

FIRST SOLUTION: Compute the Taylor series of $F(h)$ about 0 by computing the first two derivatives.

$$F(0) = \frac{mgR^2}{R^2} = mg$$

$$F'(h) = \frac{-2mgR^2}{(R+h)^3}, \quad F'(0) = \frac{-2mg}{R}$$

$$F''(h) = \frac{6mgR^2}{(R+h)^4}, \quad F''(0) = \frac{6mg}{R^2}$$

So

$$F(h) \approx F(0) + F'(0)h + \frac{F''(0)}{2}h^2 = mg - \frac{2mg}{R}h + \frac{6mg}{2R^2}h^2$$

Comparing with the give expansion we see

$$c_0 = 1, \quad c_1 = -2, \quad c_2 = 3$$

SECOND SOLUTION:

$$F(h) = \frac{mgR^2}{(R+h)^2} = \frac{mgR^2}{R^2(1+h/R)^2} = \frac{mg}{(1+h/R)^2}$$

Now use the expanion

$$(1+x)^{-2} = 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \cdots = 1 - 2x + 3x^2 + \cdots$$

So

$$F(h) = mg(1+h/R)^{-2} = mg(1 - 2\frac{h}{R} + 3\frac{h^2}{R^2} + \cdots)$$

which gives the same c_0, c_1, c_2 as before.