Math 250B (Kennedy) - Exam 2 Solutions - Spring '08

SHOW YOUR WORK. Correct answers with no work will get no credit. You may not use anything on the web other than the pplane and RK programs. There are 5 problems for a total of 100 points.

1. (22 points) Consider the first order system

$$\begin{aligned} x' &= x + y \\ y' &= 2x \end{aligned}$$

(a) Classify it as one of: stable node, unstable node, saddle, stable spiral, unstable spiral, or periodic. You can use the phase plane program to check your answer, but for full credit you must derive your answer analytically.

Solution: The second order eq for x is

$$x'' = x' + y' = x' + 2x$$

So

$$x'' - x' - 2x = 0$$

which has characteristic eq $r^2 - r - 2 = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1+8}}{2} = 2, -1$$

So its a *saddle*.

(b) Find the general solution for $x(t)$ and $y(t)$.

Solution: From part (a),

$$x(t) = c_1 e^{2t} + c_2 e^{-t}$$

So

$$y(t) = x' - x = 2c_1 e^{2t} - c_2 e^{-t} - (c_1 e^{2t} + c_2 e^{-t}) = c_1 e^{2t} - 2c_2 e^{-t}$$

2. (22 points) (a) Find the general solution of $x'' - 2x' + 5x = 0$.

Solution: The characteristic eq is $r^2 - 2r + 5 = 0$. So the roots are

$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

So the general solution of this homogeneous eq is

$$x(t) = c_1 e^t \sin(2t) + c_2 e^t \cos(2t)$$

(b) Find the general solution of $x'' - 2x' + 5x = -5t$.

Solution: We need to find one solution of the inhomogeneous eq. $-5t$ is a polynomial of degree 1, so for our "guess" we take a general polynomial of degree 1:

$$x_p(t) = A + Bt$$

Then

$$x_p' = B, \quad x'' = 0$$

So

$$x_p'' - 2x' + 5x = -2B + 5A + 5Bt$$

This needs to be $-5t$. So $-2B + 5A = 0$ and $5B = -5$. So $B = -1$ and $A = -2/5$. So the particular solution is

$$x_p = \frac{-2}{5} - t$$

and using the result of part (a) this means the general solution is

$$x = \frac{-2}{5} - t + c_1 e^t \sin(2t) + c_2 e^t \cos(2t)$$

3. (16 points total) Consider the differential equation and initial condition

$$x'' + x = e^t, \quad x(0) = 1, \quad x'(0) = -1$$

Assume that $x(t)$ has a power series about $t = 0$

$$x(t) = \sum_{n=0}^{\infty} c_n t^n = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

Find this power series to third order, i.e., find c_0, c_1, c_2, c_3. Hint: be sure you take advantage of both initial conditions. In case you forgot it, $e^t = 1 + t + t^2/2! + t^3/3! + \cdots$.

Solution:

$$x = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

$$x' = c_1 + 2c_2 t + 3c_3 t^2 + \cdots$$

$$x'' = 2c_2 + 3 \cdot 2c_3 t + \cdots$$

First look at the intial conditions. $x(0) = c_0$, so $c_0 = 1$. $x'(0) = c_1$, so $c_1 = -1$. Now look at the dif eq:

$$2c_2 + 3 \cdot 2c_3 t + \cdots + c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots = 1 + t + t^2/2! + t^3/3! + \cdots$$

Now we look at terms of different orders in t. The t^0, i.e., the constant terms give

$$2c_2 + c_0 = 1$$

Since $c_0 = 1$, this gives $c_2 = 0$. The t^1 terms give

$$6c_3 + c_1 = 1$$

Since $c_1 = -1$, this gives $c_3 = 1/3$. Note that we don't need to go to any higher order in the dif. eq. So

$$x(t) = 1 - t + 0t^2 + \frac{1}{3}t^3 + \cdots$$

4. (20 points) (a) In the following system a is a parameter. Determine the value(s) of a for which the solutions are periodic. You can use the phase plane program to check your answer, but for full credit you must derive your answer analytically.

$$\begin{aligned} x' &= y \\ y' &= -x + ay \end{aligned}$$

Solution: Convert it to a second order eq in x:

$$x'' = y' = -x + ay = -x + ax'$$

So $x'' - ax' + x = 0$, which has characteristic eq. $r^2 - ar + 1 = 0$. The roots are

$$r = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

You get periodic solutions when the roots are complex with zero real part. So $a = 0$.

(b) Find the equilibrium point of the following. (It will depend on a.)

$$\begin{aligned} x' &= y + 1 \\ y' &= -x + ay \end{aligned}$$

Solution: Set $x' = 0$, $y' = 0$.

$$\begin{aligned} 0 &= y + 1 \\ 0 &= -x + ay \end{aligned}$$

So $y = -1$ and $x = -a$. So there is one equilibrium solution at $(-a, -1)$.

5. (20 points) A nonlinear pendulum is described by the differential equation

$$x'' + 4\sin(x) = 0$$

We start the pendulum with $x(0) = \pi/2$, $x'(0) = 0$.

(a) Find the corresponding system of first order differential equations.

Solution: As always we introduce a new function $y(t)$ by $y = x'$. Then $y' = x'' = -4\sin(x)$. So the system is

$$\begin{aligned} x' &= y \\ y' &= -4\sin(x) \end{aligned}$$

(b) Use the phase plane program to estimate the velocity of pendulum when it is vertical, i.e., $x = 0$. You can just give a number for your answer without explanation.

Solution: Use the initial condition $(x, y) = (\pi/2, 0)$, and see where the trajectory is when $x = 0$. You should find a y around -2.8.

(c) By finding the trajectory for this solution, determine exactly the velocity of the pendulum when $x = 0$.

Solution:

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{-4\sin(x)}{y}$$

Solve this by separation of variables.

$$y\,dy = -4\sin(x)\,dx$$

$$\frac{1}{2}y^2 = 4\cos(x) + C$$

We have the initial condition $x = \pi/2$, $y = 0$, so

$$\frac{1}{2}0^2 = 4\cos(\pi/2) + C$$

which gives $C = 0$. So our trajectory is

$$\frac{1}{2}y^2 = 4\cos(x)$$

At $x = 0$ this gives $y = \pm\sqrt{8} = \pm 2\sqrt{2}$. The angle of the pendulum is decreaing, so $y = -2\sqrt{2}$.