Math 250B (Kennedy) - Exam 3 - Spring '08

SHOW YOUR WORK. Correct answers with no work will get no credit. You may not use anything on the web other than the P-Plane programs. There are 5 problems for a total of 100 points.

1. (20 points) Consider the homogeneous and inhomogeneous differential equations

$$(t-1)x'' - tx' + x = 0 \tag{1}$$
$$(t-1)x'' - tx' + x = 2te^{-t} \tag{2}$$

(a) Each of the three functions below satisfies one of the above two equations. Determine which equation, (1) or (2), each function satisfies.

$x_1(t) = e^{-t}$ satisfies dif. eq. (2)

$x_2(t) = e^{t}$ satisfies dif. eq. (1)

$x_3(t) = t$ satisfies dif. eq. (1)

(b) Find the general solution of the homogeneous equation (1).
Solution: $c_1 e^t + c_2 t$.

(c) Find the general solution of the inhomogeneous equation (2).
Solution: $e^{-t} + c_1 e^t + c_2 t$.

2. (16 points) One solution of the differential equation below is $x_1(t) = t$. Use reduction of order to find a second solution of this differential equation.

$$t^2 x'' - t(t+2)x' + (t+2)x = 0$$

Solution: Let $x = tz$. Then $x' = tz' + z$, $x'' = tz'' + 2z'$. So the dif. eq. becomes

$$t^2(tz'' + 2z') - t(t+2)(tz' + z) + (t+2)tz = 0$$

which simplifies to

$$t^3 z'' - t^3 z' = 0$$

or just $z'' - z' = 0$. Letting, $u = z'$, this is $u' = u$. You can use separation or variables, an integrating factor or just guess that the solution is $u = e^t$. So $z = e^t$. So $x = te^t$.

3. (12 points) Initially, container A contains 20 liters of pure water, and container B contains 10 liters of pure water. Polluted water flows into container A at the rate of 2 liters/min. It has a pollution concentration of 5 gm/liter. Water flows out of container A at the rate of 2 liters/min. Water flows from A to B at the rate of 1 liters/min and from B to A at the rate of 1 liters/min.
(a) Let $x(t)$ and $y(t)$ be the **concentrations** of pollutant in containers A and B, respectively. Find the differential equations and initial conditions for $x(t), y(t)$. (Don't solve them!)

Solution:

$$\begin{aligned} x' &= \frac{10}{20} - \frac{3x}{20} + \frac{y}{20} \\ y' &= \frac{x}{10} - \frac{y}{10} \end{aligned}$$

(b) Find the equilibrium solution. (You can do this without doing part a.)

Solution: Setting $x' = 0$ and $y' = 0$,

$$\begin{aligned} 0 &= \frac{10}{20} - \frac{3x}{20} + \frac{y}{20} \\ 0 &= \frac{x}{10} - \frac{y}{10} \end{aligned}$$

and the solution is $x = y = 5gm/l$. Note that you can see this must be the equilibrium directly from the model. After a long time the water in both containers will have the same concentration as the polluted water flowing in, i.e., $5gm/l$.

4. (30 points) Consider the system

$$\begin{aligned} x' &= 2x - y \\ y' &= -6x + y \end{aligned}$$

(a) Find the general solution.
Solution: We use the linear algebra approach. The matrix is

$$\begin{pmatrix} 2 & -1 \\ -6 & 1 \end{pmatrix}$$

Roots are given

$$0 = det \begin{pmatrix} 2-r & -1 \\ -6 & 1-r \end{pmatrix} = r^2 - 3r - 4 = (r-4)(r+1)$$

So the roots are 4 and -1.

For $r = 4$ the eigenvector equation is

$$\begin{aligned} 2x - y &= 4x \\ -6x + y &= 4y \end{aligned}$$

These both simplify to $2x + y = 0$. So one eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

For $r = -1$ the eigenvector equation is

$$\begin{aligned} 2x - y &= -x \\ -6x + y &= -y \end{aligned}$$

These both simplify to $3x - y = 0$. So one eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

So the general solution is

$$c_1 e^{4t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

or

$$\begin{aligned} x(t) &= c_1 e^{4t} + c_2 e^{-t} \\ y(t) &= -2c_1 e^{4t} + 3c_2 e^{-t} \end{aligned}$$

(b) Find all the straight line orbits (or trajectories).

Solution: Using our work in (a), the linear orbits are $y = -2x$ and $y = 3x$.

(c) Find all the straight line orbits for

$$\begin{aligned} x' &= 2x - y + 1 \\ y' &= -6x + y - 5 \end{aligned}$$

Solution: Note that this system is the previous one plus some constants. Start by finding the equilibrium of the new system:

$$\begin{aligned} 0 &= 2x_0 - y_0 + 1 \\ 0 &= -6x_0 + y_0 - 5 \end{aligned}$$

Add these to get $0 = -4x_0 - 4$. So $x_0 = -1$. So $y_0 = -1$. So the equilibrium is $(-1,-1)$. The solutions for the new system are just the solutions for original system plus $(-1,-1)$. In particular the linear trajectories for the new system are lines with slopes -2 and 3 passing through $(-1,-1)$. So $y+1 = -2(x+1)$ and $y+1 = 3(x+1)$.

5. (22 points) For the non-linear autonomous system

$$\begin{aligned} x' &= y^2 - 1 \\ y' &= x\,e^y \end{aligned}$$

(a) Find all the equilibrium points.

Solution: We have from the first equation $y = \pm 1$ and from the second $x = 0$ since e^y is never 0. So there are two equilibria: $(0,1)$ and $(0,-1)$.

(b) For **one** of the equilibria, find the linear system that approximate the original system near the equilibrium. (You get to choose which equilibrium.)

Solution: The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 2y \\ e^y & xe^y \end{pmatrix}$$

At $(0,1)$ we have

$$J(0,1) = \begin{pmatrix} 0 & 2 \\ e & 0 \end{pmatrix}$$

So the linearization is

$$\begin{aligned} x' &= 2(y-1) \\ y' &= ex \end{aligned}$$

At $(0,-1)$ we have

$$J(0,1) = \begin{pmatrix} 0 & -2 \\ e^{-1} & 0 \end{pmatrix}$$

So the linearization is

$$\begin{aligned} x' &= -2(y+1) \\ y' &= e^{-1}x \end{aligned}$$

(c) State what you can conclude from the linearization theorem for your equilibrium in the original system.

Solution:

At $(0, 1)$ the roots are given by

$$0 = det \begin{pmatrix} -r & 2 \\ e & -r \end{pmatrix} = r^2 - 2e$$

So the roots are $\pm\sqrt{2e}$. So the linear system has a saddle and the linearization theorem says the original system does too at this equilibrium.

At $(0, -1)$ the roots are given by

$$0 = det \begin{pmatrix} -r & -2 \\ e^{-1} & -r \end{pmatrix} = r^2 + 2e^{-1}$$

So the roots are $\pm i\sqrt{2/e}$. So the linear system has a center (periodic solutions). But the linearization theorem only says that the original system can have a center, stable spiral or unstable spiral at this equilibrium.