A Counting

A.1 First principles

If the sample space Ω is finite and the outomes are equally likely, then the probability measure is given by $P(E) = |E|/|\Omega|$ where $|E|$ denotes the number of outcomes in the event E. So to compute probabilities in this setting we need to be able to count things.

There are two basic principles:

Addition principle: If I have m forks and k knives, then I have $m+k$ ways to choose a fork *or* a knife.

Multiplication principle: I have have m forks and k knives then there are mk ways to pick a fork *and* a knife.

Notation:

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 \tag{1}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)\cdots 3 \cdot 2 \cdot 1} \tag{2}$$

Examples: Using the letters A,B,C,D,E,F,G, how many four letter "words" can I form if

(a) I can repeat a letter. **Solution:** There are 7 letters. By the multiplication principle, the number of words is $7 \cdot 7 \cdot 7 \cdot 7 = 7^4$.

(b) I cannot repeat a letter. **Solution:** Again, the multiplcation principle gives the answer: $7 \cdot 6 \cdot 5 \cdot 4$.

(c) The word must begin or end with an A and I cannot repeat a letter. **Solution:** We start with the addition principle. The word must either begin with an A or end with an A. (It can't do both since reps are not allowed.) The number of words that begin with an A is $1 \cdot 6 \cdot 5 \cdot 4$. The number of words that end with an A is $6 \cdot 5 \cdot 4 \cdot 1$. Adding these two types of words, the answer is $2 \cdot 6 \cdot 5 \cdot 4$.

(d) The word must begin or end with an A and I can repeat letters. **Solution:** Since we can repeat letters, the word could both begin and end with an A. We must be careful to avoid overcounting. There are three types of words: AXXN,NXXA and AXXA where X stands for any letter and N stand for any letter other than A. Since reps are allowed, for each X there are 7 choices and for each N there are 6 choices. So the number of words of type AXXN is $6^2 \cdot 7$. The number of type XXXA is the same, and the number of type AXXA is 7^2. So the answer is $2 \cdot 6^2 \cdot 7 + 7^2$.

In any counting problem we should ask the following questions

1. Does order matter?
2. Are repetitions allowed?
3. Are objects indentical or distinguishable?

We will consider four counting problems:

P1: We have n different objects. How many ways are there to arrange k of them in a row ? (Repetitions are not allowed.)
P2: We have n objects, not necessarily different. How many ways to arrange **all** of them in a row?
P3: We have n different objects. How many ways to pick a set of k of them? (The order of the k chosen does not matter.)
P4: We have n different types of objects with an unlimited number of each type. How many ways to pick a set of k objects? (The order does not matter.)

A.2 Permutations

In this section we consider counting problems P1 and P2.

Theorem 1 *Given n distinct objects, the number of ways to place k of them in a row without repetitions is*

$$\frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$$

Proof: This follows immediately from the multiplication principle. ∎

Example: In the poker game of 5 card draw, I am dealt five cards. (A deck of cards has 52 cards divided into 4 different suits. Each suit contains one ace, 2,3,4,5,6,7,8,9,10,jack,queen and king)
(a) What is the probability I get one ace?
(b) What is the probability I get a flush? (A flush means that all five cards are of the same suit.)
Solution: We take the sample space to be all ways to arrange 5 of the 52 cards in a row. So

$$|\Omega| = 52\cdots 51\cdots 50\cdots 49\cdots 48$$

Now consider the hands that contain one ace. Let A denote an ace, X a card other than an ace. Then the possibilities are AXXXX,XAXXX,XXAXX,XXXAX and XXXXA. There are 4 aces and 48 non-aces. So for each possibility the number of hands is $4\cdot 48\cdot 47\cdot 46\cdot 46$. So the total number of hands with one ace is $5\cdot 4\cdot 48\cdot 47\cdot 46\cdot 45$. Thus the probability is

$$\frac{5\cdot 4\cdot 48\cdot 47\cdot 46\cdot 45}{52\cdots 51\cdots 50\cdots 49\cdots 48} = \frac{3243}{10829} \approx 29.9\% \tag{3}$$

Now consider a flush. We start by choosing a suit. There are 4 choices. Then we must take the 5 cards from the 13 in the suit. So there are $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$ choices. So the probability is

$$\frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdots 51 \cdots 50 \cdots 49 \cdots 48} = \frac{99}{49980} \approx 0.198\% \tag{4}$$

In the above example we defined the sample space as if the order of the five cards was important. There is another way to work this problem in which the sample space is just the possible hands without considering the order of the cards. We will return to this approach in the next section.

Sampling terminology I have n balls numbered 1 to n in a hat. I draw a ball, note its number, put it back. I repeat this process a total of k times. This is called *sampling with replacement*. In this case $|\Omega| = n^k$. If we do not replace the ball after it is drawn we call it *sampling without replacement*. In this case

$$|\Omega| = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

We now consider the counting problem $P2$. We have n objects but they are not all different. How many ways to arrange all of them in a row?

Example: I have 5 A's, 3 B's and 1 C. How many ways to arrange all 9 letters? Let N be the answer. We find N by first doing a different problem. Suppose we add subscripts to distiguish the letter of the same type. So we have $A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, C$ and we ask how many ways to arrange them? This is easy; it is just $(5+3+1)! = 9!$. Now we count the same thing in a different by a more complicated two stage process. First we arrange the original 5 A's, 3 B's and 1 C. There are N ways to do this. Now we add the subscripts $1, 2, 3, 4, 5$ to the A's. There are 5! ways to do this. Then we add the subscripts $1, 2, 3$ to the B's. There are 3! ways to do this. Finally we add the 1 to the C; there is only $1 = 1!$ way to do this. So the number of ways to add all the subscripts is 5!3!1!. So the second solution gives $N5!3!1!$. Of course, both solution should give the same answer, so

$$9! = N5!3!1!$$

Solving for N we have

$$N = \frac{9!}{5!3!1!}$$

We generalize this as a theorem.

Theorem 2 *Suppose we have r types of objects. We have n_j of type j. Let $n = \sum_{j=1}^{r} n_j$ be the total number of objects. Then the number of ways to arrange all n objects in a line is*

$$M_n(n_1, \cdots, n_r) = \frac{n!}{\prod_{j=1}^{r} n_j!}$$

Remark: Suppose we only want to arrange k of the above n objects where $k < n$. This is a much harder counting problem and there is no simple formula.

Proof: Consider a different problem. Add labels $1, 2, \cdots, n_j$ to the objects of type j so we can tell them apart. Then there are simply $n!$ ways to arrange all of them. We can count this in a more complicated way by first arranging the original objects and then adding labels. There are $M_n(n_1, \cdots, n_r)$ ways to arrange the original objects. The number of ways to add labels to the objects of type j is $n_j!$. So the total number of ways to add labels to all of them is $\prod_{j=1}^{n} n_j!$. The two solutions must give the same answer, so

$$n! = M_n(n_1, \cdots, n_r) \prod_{j=1}^{n} n_j!$$

The equation in the theorem follows. ∎

Terminology: The number $M_n(n_1, \cdots, n_r)$ is called a "multinomial coefficient." If $r = 2$, we have

$$M_n(n_1, n_2) = \frac{(n_1 + n_2)!}{n_1! n_2!} = \binom{n_1 + n_2}{n_1} = \binom{n_1 + n_2}{n_2}$$

which is called a binomial coefficient.

Example: A car dealer has 20 parking spaces. He has 15 new cars to put in them. Of these 15, 8 are Accords, 5 are Civics and 2 are Elements. How many ways to park the cars? (Assume that the cars of the same model are identical).

Solution: If there were 15 parking places this solution would simply be

$$\frac{15!}{8!5!2!}$$

For the given problem, imagine that we have 5 invisible cars for a total of twenty cars. Then there are

$$\frac{20!}{5!8!5!2!}$$

ways to park them.

Example: I have 10 identical balls and 5 different urns. I am going to put each ball into one of the urns.

(a) How many ways if there are no restrictions? In particular an urn can be empty and there is no limit to the number in an urn.
(b) How many ways if we add the restriction that each urn must contain at least one ball?

Solution: For (a) we will turn it into a word problem. The urns are different so we can label them 1,2,3,4,5. Line them up in order. We then represent a choice of how to put the balls in by a word with 10 "B"'s and 4 "X" 's. The X's mark the boundary between two urns. So the word BBXBBBBBXBBXXB means we put 2 balls into the first urn, 5 into the second urn, 2 into the third urn, none into the fourth urn and 1 into the fifth urn. Conversely, if we put 3 balls into the first urn, none into the second urn, 4 into the third urn, 1 into the fourth urn and 2 into the fifth urn, then the word is BBBXXBBBBXBXBB. There is a one to one correspondence between ways of putting the balls into the urns and words with 10 B's and 4 X's. Note that the number of X's is one less than the number of urns since there are only 4 "boundaries" between the urns. You may be tempted to add an X at the start and end of your word. Why is this a bad idea? Now we have a simple word counting problem. The number of words with 10 B's and 4 X's is

$$\frac{14!}{10!4!}$$

For part (b), we can start by putting one ball into each urn. Then we are left with 5 balls to put into the urns with no constraints. This is the same as the part (a) with 5 balls instead of 10. So the answer is

$$\frac{9!}{5!4!}$$

The method used in the above example proves the following theorem. As we will see at the end of the next section, this is really counting problem P4 in disguise.

Theorem 3 *Given r identical objects and n different urns, the number of ways to put the objects into the urns with no constraint except that all the objects must be placed is*

$$\frac{(r+n-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

A.3 Combinations

We now consider counting problem P3. We have n different objects. How many ways to pick a set of k of them if the order of the k chosen does not matter.

Theorem 4 *The number of ways to pick a* **set** *of k objects from n different objects is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof: We will prove a more general version of this theorem later. ∎
Example: Suppose we draw 5 cards from a deck.
(a) What is the probability of a flush? (All cards of the same suit.)
(b) What is the probability of two pair? (A pair means two cards with the same number.)

Solution: In both parts we take the sample space to be all *subsets* with 5 cards. We do not care about the order of the five cards. So

$$|\Omega| = \binom{52}{5}$$

(a) First we pick a suit (4 choices). Then we pick 5 cards from that suit. So the probability is

$$\frac{4\binom{13}{5}}{\binom{52}{5}} \approx 0.2\%$$

(b) First we choose the "numbers" for the two pairs. This amounts to choosing a subset of 2 from 13. (The order does not matter. A pair of 2's and a pair of jacks is the same as a pair of jacks and a pair of 2's.) Then for each of the two numbers we choose two suits. So the probability is

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}}{\binom{52}{5}} \approx 4.8\%$$

Example: A hat has 10 one dollar bills, 12 five dollar bills. I draw 8 at random (without replacement). What is the probability I get 6 one dollar bills and 2 five dollar bills?
Solution:

$$\frac{\binom{10}{6}\binom{12}{2}}{\binom{22}{8}}$$

The above example generalizes.

Theorem 5 *A hat contains balls with m different colors. There are r_i balls of the ith color, $i = 1, 2, \cdots, m$. We draw n balls without replacement. Let X_i be the number of balls of color i. Then*

$$P(X_1 = k_1, X_2 = k_2, \cdots X_m = k_m) = \frac{\prod_{i=1}^{m} \binom{r_i}{k_i}}{\binom{N}{n}}$$

where N is the total number of balls in the hat, i.e, $N = r_1 + r_2 \cdots r_m$. (Of course the k_i must sum to n.)

Picking a set of r from a set of n is equivalent to dividing the objects into two groups. One group is the set of chosen objects, the other group is the set of those not chosen. So the number of ways to divide a set of r into two distinguishable groups with n_1 in the first group and n_2 in the second group is

$$\binom{n}{n_1} = \frac{n!}{n_1!n_2!}$$

assuming of course that $n_1 + n_2 = n$. This generalizes to more than two groups:

Theorem 6 *Given n different objects, the number of ways to divide them into r different groups with n_i in the ith group is*

$$\frac{n!}{\prod_{i=1}^{r} n_i!}$$

(We are assuming the order within a group does not matter.)

Proof: Let N be the answer to the above problem. Consider a different counting problem - the number of ways of arranging all n in a row. Of course there are $n!$ ways to do this. Now consider doing it in two stages. First divide them into groups $1, 2, \cdots, r$ with r_i in group i. Then arrange the objects in each group in linear order. For group i there are $n_i!$ ways to do this. These two stages give a linear order for all the objects: we put group 1 on the left, arranged in its chosen order, then group 2, arranged in its chosen order, ..., and finally group r on the right arranged in its chosen order. The two ways of counting the number of arrangements of all n must agree, so

$$n! = N \prod_{i=1}^{r} n_i!$$

Solving for N proves the theorem. ∎

The following is obvious if you think about it in the right way.

Theorem 7 *If we have r types of objects with n_i of type i, then the number of ways to pick a subset of any size (including the empty set) is*

$$\prod_{i=1}^{r} (n_i + 1)$$

Proof: The subset is completely determined by specifying how many of each type we have. For type i we can take anywhere from 0 to n_i of them. So by the multiplication principle the answer is the product given in the theorem. ∎

The last theorem of the previous section can be reformulated as a combination problem. Note that this solves counting problem P4.

Theorem 8 *If we have n types of objects and an unlimited number of each type, the number of ways to choose a subset of k is*

$$\binom{n+k+1}{k}$$

Proof: Think of the n types as n urns. Initially we have k identical objects. We assign them types by putting them into the urns. ∎

Example: Consider expanding out $(x+y+z)^{10}$. What is the coefficient of $x^3y^5z^2$?
Solution: The expansion has 3^{10} terms. We can think of a term as a 10 letter word (using only x,y and z). The word $xxyzyzzyzx$ means we took x from the first factor of $(x+y+z)$, x from the second factor, y from the third, z from the fourth, ... The coef of $x^3y^5z^2$ will be the number of words with exactly 3 x's, 5 y's and 2 z's. This is problem P2 and the answer is

$$\frac{10!}{3!\,5!\,2!}$$

Example: I have 20 dollars. In the Dollar Store, everything costs one dollar. There are 8 items in the store that I like. In how many ways can I spend all of my money if I only buy things I like and there are no constraints on how many of a particular item I buy? Assume the store has at least 8 of every item it carries.
Solution: This is counting problem P4:

$$\frac{(20+8-1)!}{20!(8-1)!} = \frac{27!}{20!7!} = \binom{27}{20}$$

Example: I deal five cards from the usual 52 card deck in a row. (I keep them in the order in which they were dealt.) Find the probability that
(a) They are all the same suit.
(b) They are all the same suit and the ranks are in increasing order.
(c) All of the red cards are left of all of the black cards. (This includes the cases that all the card are red or all the cards are black.)
Solution: Since order matters, the sample space is the number of ways to arrange 5 things from 52 in a line. Thus

$$|\Omega| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!}$$

In each part below the probability will be the number in the event divided by $|\Omega|$.
(a) There are 4 choices of suit. Given the suit there are 13 cards in it and we must put 5 of them in a line. So number of outcomes in the event is

$$4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 4\frac{13!}{8!}$$

(b) Again there are 4 choices of suit. Given a suit we choose a subset of 5 of the 13 cards. Since they must be arranged in increasing order, this determines a unique outcome. So the number of outcomes in the event is

$$4 \cdot \binom{13}{5}$$

(c) We break the event up into cases based on the sequence of colors of the cards. There are six cases: RRRRR,RRRRB,RRRBB,RRBBB,RBBBB,BBBBB. The number of outcomes in each is

$$\begin{aligned}
|RRRRR| &= 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \\
|RRRRB| &= 26 \cdot 25 \cdot 24 \cdot 23 \cdot 26 \\
|RRRBB| &= 26 \cdot 25 \cdot 24 \cdot 26 \cdot 25 \\
|RRBBB| &= 26 \cdot 25 \cdot 26 \cdot 25 \cdot 24 \\
|RBBBB| &= 26 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \\
|BBBBB| &= 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22
\end{aligned}$$

The total number of outcomes in the event is the sum of these six products.