Math 466/566 - Homework 5 Solutions


Solution: The expected value of the sample mean is always the population mean, so the sample mean is always an unbiased estimator. The variance of a Poisson RV is equal to its mean, \( \theta \). So the variance of the sample mean is \( \theta/n \). To find the Cramer-Rao bound we must compute \( I(\theta) \).

\[
f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}
\]

where \( x = 0, 1, 2, 3, \ldots \). So

\[
\ln(f(x|\theta)) = x \ln(\theta) - \theta - \ln(x!)
\]

\[
\frac{\partial \ln(f(x|\theta))}{\partial \theta} = \frac{x}{\theta} - 1
\]

\[
\frac{\partial^2 \ln(f(x|\theta))}{\partial \theta^2} = -\frac{x}{\theta^2}
\]

Since the Poisson RV is discrete, \( I(\theta) \) is given by a sum, not an integral

\[
I(\theta) = \sum_{x=0}^{\infty} \frac{x}{\theta^2} f(x|\theta) = \frac{1}{\theta^2} \sum_{x=0}^{\infty} x f(x|\theta)
\]

Note that the last sum is just the expected value of \( X \) and so is \( \theta \). So \( I(\theta) = 1/\theta \). So Cramer-Rao says the variance of an unbiased estimator is at least \( \theta/n \). So the sample mean has minimal variance.


\[
\frac{\partial^2 \ln(f(x|\theta))}{\partial \theta^2} = 2 \frac{(x - \theta)^2 - s^2}{(s^2 + (x - \theta)^2)^2}
\]

So

\[
I(\theta) = \int_{-\infty}^{\infty} 2 \frac{(x - \theta)^2 - s^2}{(s^2 + (x - \theta)^2)^2} f(x|\theta) dx = \frac{2s}{\pi} \int_{-\infty}^{\infty} \frac{(x - \theta)^2 - s^2}{(s^2 + (x - \theta)^2)^3} dx
\]
A nasty integral, but we can simplify it a bit. A simple change of variables \( x \rightarrow x + \theta \) shows \[ I(\theta) = \frac{2s}{\pi} \int_{-\infty}^{\infty} \frac{x^2 - s^2}{(s^2 + x^2)^3} \, dx \]

Then another change of variables \((x \rightarrow sx)\) shows \[ I(\theta) = \frac{2}{\pi s^2} \int_{-\infty}^{\infty} \frac{x^2 - 1}{(1 + x^2)^3} \, dx \]

You can do the integral using tables, a software package or even contour integration if you’ve taken complex variables. I think you get \( I(\theta) = 1/(2s^2) \).

So Cramer Rao says the variance of any unbiased estimator is at least \( 2s^2/n \).

3. Consider the exponential distribution \( f(x|\theta) = \theta e^{-\theta x} \) where \( \theta > 0 \). As always, we have a random independent sample \( X_1, X_2, X_3, \cdots, X_n \). The mean of this distribution is \( \mu = 1/\theta \).

(a) Find the maximum likelihood estimators of the mean \( \mu \) and of \( \theta \).

Solution:

\[ f(x_1, x_2, \cdots, x_n) = \theta^n \exp(-\theta \sum_{i=1}^{n} x_i) \]

So

\[ \ln(f(x_1, x_2, \cdots, x_n)) = n \ln(\theta) - \theta \sum_{i=1}^{n} x_i \]

Take derivative with respect to \( \theta \) and set it to zero to find the maximum:

\[ \frac{n}{\hat{\theta}} - \sum_{i=1}^{n} x_i = 0 \]

So the MLE for \( \theta \) is

\[ \hat{\theta} = \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^{-1} = \frac{1}{\bar{X}_n} \]
Since MLE’s satisfy the principle of functional invariance, the MLE of $\mu = 1/\theta$ is

$$\hat{\mu} = \bar{X}_n$$

(b) By appealing to a theorem, show that for large $n$, the MLE for $\theta$ is approximately normal, with mean $\theta$ and variance $\theta^2/n$.

**Solution:** We use theorem 8.5 in the book. It says that $\hat{\theta}$ is approximately normal with mean $\theta$ and variance $[nI(\theta)]^{-1}$. To compute $I(\theta)$,

$$I(\theta) = -\int \frac{\partial^2 \ln(f(x|\theta))}{\partial \theta^2} f(x|\theta) \, dx = \int \frac{1}{\theta^2} f(x|\theta) \, dx = \frac{1}{\theta^2}$$

So the variance is approximately $\theta^2/n$.

4. Consider the geometric density $f(x|p) = p(1 - p)^x$ where $x = 0, 1, 2, \ldots$. We have a random independent sample $X_1, X_2, X_3, \ldots, X_n$. Find the maximum likelihood estimator of the mean and of $p$.

**Solution:**

$$f(x_1, x_2, \ldots, x_n) = p^n (1 - p)^{\sum_{i=1}^{n} x_i}$$

$$\ln(f(x_1, x_2, \ldots, x_n)) = n \ln(p) + \sum_{i=1}^{n} x_i \ln(1 - p)$$

Take derivative with respect to $p$ and set it to zero to find the maximum:

$$\frac{1}{\hat{p}} - \sum_{i=1}^{n} x_i \frac{1}{1 - \hat{p}} = 0$$

Solving for $\hat{p}$, we find the MLE for $p$ is

$$\hat{p} = \frac{1}{1 + \bar{X}_n}$$

The mean of the geometric distribution is given by $\mu = (1 - p)/p$. So by functional invariance, the MLE for the mean is

$$\hat{\mu} = \frac{1 - \hat{p}}{\hat{p}} = \bar{X}_n$$
5. Consider the uniform distribution on \([0, \theta]\). We have a random sample \(X_1, X_2, \ldots, X_n\).

(a) Find the maximum likelihood estimator of \(\theta\). Hint: don’t use derivatives. Just try to maximize the likelihood given \(X_1, \ldots, X_n\).

**Solution:** The likelihood function is \(\theta^{-n}\) when \(x_1, x_2, \ldots, x_n\) all belong to \([0, \theta]\). Otherwise it is zero. So we can write it as

\[
f(x_1, \ldots, x_n|\theta) = \theta^{-n} 1(x_i \leq \theta, i = 1, \ldots, n) = \theta^{-n} 1(\max x_i \leq \theta)
\]

We want to maximize this as a function of \(\theta\). This is equivalent to maximizing \(\theta^{-n}\) subject to the constraint \(\max x_i \leq \theta\). The max occurs at \(\hat{\theta} = \max x_i\). So \(\hat{\theta}\) is \(X_{(n)}\), the largest order statistic.

(b) Find the MLE of the mean \(\mu = \theta/2\).

**Solution:** By the principal of functional invariance, the MLE of the mean is \(\hat{\mu} = \bar{X}_{(n)}/2\).

(c) (566 only) Now suppose that we have the uniform distribution on \([\theta_1, \theta_2]\) with both \(\theta_1\) and \(\theta_2\) unknown. Find the MLE’s of \(\theta_1\) and \(\theta_2\) and of the mean \(\mu = (\theta_1 + \theta_2)/2\).

**Solution:** Now we must maximize \((\theta_2 - \theta_1)^{-n}\) as a function of \(\theta_1\) and \(\theta_2\) subject to the constraints \(\theta_1 \leq \min x_i\) and \(\theta_2 \geq \max x_i\). The max occurs at

\[
\hat{\theta}_1 = X_{(1)}, \quad \hat{\theta}_2 = X_{(n)}
\]

By functional invariance, the MLE of \(\mu\) is \((X_{(1)} + X_{(n)})/2\).