Math 466/566 - Homework 6

1. Book, chapter 10, number 3. Feel free to use results from class. This should make the problem quite easy. Note that the uniform distribution on $[0,1]$ is a special case of the gamma distribution.

2. The number of defects in a magnetic tape has a Poisson distribution with unknown mean θ. The prior distribution of θ is a gamma distribution with $\alpha = 3, \beta = 1$. Five rolls of magnetic tape are tested for defects and it is found that the number of defects is $2, 2, 6, 0, 3$. If we use the squared error loss function, what is the Bayes estimate of θ ?

3. Heights of individuals in a population have a normal distribution with unknown mean θ and standard deviation of 2. The prior distribution of θ is normal with mean 68 inches and standard deviation of 1 inch. 10 people are selected from the population at random and their average height is found to be 69.5 inches.
(a) If the squared error loss function is used, what is the Bayes estimate of θ?
(a) If the absolute error loss function ($L(\theta, a) = |\theta - a|$) is used, what is the Bayes estimate of θ?

4. Suppose that the population has a Poisson distribution with mean θ which is unknown. Suppose that the prior distribution of θ is a gamma distribution with parameters α and β.

(a) Show that the posterior distribution of θ is again a gamma distribution with parameters

$$\alpha' = \alpha + n\bar{X}_n, \quad \beta' = \beta + n \tag{1}$$

(As always, n is the sample size and $\bar{X}_n$ is the sample mean.)

(b) What is the Bayes estimator (using squared error loss) for θ?

(c) What is the limit as $n \to \infty$ of the Bayes estimator?

5. Suppose that a random sample is to be taken from a normal distribution with unknown mean θ and standard deviation 2. The prior distribution of θ is normal with mean μ_0 and standard deviation 1.
(a) What is the smallest sample size that will insure the standard deviation of the posterior distribution of θ is at most 0.1 ?

(b) Suppose we use the squared error loss function. In general, the risk depends on θ. Show that in this setting it does not, and find the risk for the sample size you found in part (a).

6. **(566 students only)** Let $c > 0$ and

$$L(\theta, a) = \begin{cases} c\,|\theta - a| & \text{if } \theta < a \\ |\theta - a| & \text{if } \theta \geq a \end{cases} \tag{2}$$

Assume that θ has a continuous distribution. Show that the Bayes estimator of θ is the $1/(1+c)$ quantile of the posterior distribution of θ. **Hint:** You already done this in a previous homework for $c = 1$, in which case the estimator is the median. Mimic this proof.