

Math 466/566 - Homework 6 - Solutions

1. Book, chapter 10, number 3.

Solution:

$$f(x|\theta) = \theta^{-1}1(0 \leq x \leq \theta)$$

and

$$\pi(\theta) = \beta^2 \theta e^{-\beta\theta}$$

So the joint density of x and θ is

$$f(x, \theta) = 1(0 \leq x \leq \theta) \beta^2 e^{-\beta\theta}$$

We compute $f(x)$ by integrating out θ . So

$$\begin{aligned} f(x) &= \int 1(0 \leq x \leq \theta) \beta^2 e^{-\beta\theta} d\theta \\ &= \int_x^\infty \beta^2 e^{-\beta\theta} d\theta \\ &= \beta e^{-\beta x} \end{aligned} \tag{1}$$

Thus

$$\pi(\theta|x) = \frac{f(x, \theta)}{f(x)} = 1(0 \leq x \leq \theta) \beta e^{-\beta(\theta-x)}$$

The mean of the posterior is

$$\begin{aligned} E[\Theta|x] &= \int \theta 1(0 \leq x \leq \theta) \beta e^{-\beta(\theta-x)} d\theta \\ &= \int_x^\infty \theta \beta e^{-\beta(\theta-x)} d\theta \\ &= \int_0^\infty (\theta + x) \beta e^{-\beta\theta} d\theta \\ &= \int_0^\infty (\theta + x) \beta e^{-\beta\theta} d\theta \\ &= \frac{1}{\beta} + x \end{aligned} \tag{2}$$

2. The number of defects in a magnetic tape has a Poisson distribution with unknown mean θ . The prior distribution of θ is a gamma distribution with $\alpha = 3, \beta = 1$. Five rolls of magnetic tape are tested for defects and it is found that the number of defects is 2, 2, 6, 0, 3. If we use the squared error loss function, what is the Bayes estimate of θ ?

Solution: The posterior distribution is a gamma distribution with $\alpha' = \alpha + n\bar{X}_n$ and $\beta' = \beta + n$. We have $n = 5, \bar{X}_n = 13/5$. So $\alpha' = 13 + 3 = 16$ and $\beta' = 1 + 5 = 6$. With squared error loss, the Bayes estimator is the mean of the posterior distribution. The mean of the gamma distribution is α'/β' , so the Bayes estimator is $16/6 = 8/3$.

3. Heights of individuals in a population have a normal distribution with unknown mean θ and standard deviation of 2. The prior distribution of θ is normal with mean 68 inches and standard deviation of 1 inch. 10 people are selected from the population at random and their average height is found to be 69.5 inches.

(a) If the squared error loss function is used, what is the Bayes estimate of θ ?

Solution: The posterior distribution is normal with mean

$$\frac{\frac{\sigma^2}{n}\mu_0 + \alpha^2\bar{X}_n}{\frac{\sigma^2}{n} + \alpha^2} = 69.07$$

The Bayes estimator is the mean of the posterior, i.e., 69.07.

(a) If the absolute error loss function ($L(\theta, a) = |\theta - a|$) is used, what is the Bayes estimate of θ ?

Solution: With absolute error the Bayes estimator is the median of the posterior distribution. But for normal distributions the mean and median are equal, so the Bayes estimator with absolute error loss function is also 69.07.

4. Suppose that the population has a Poisson distribution with mean θ which is unknown. Suppose that the prior distribution of θ is a gamma distribution with parameters α and β .

(a) Show that the posterior distribution of θ is again a gamma distribution with parameters

$$\alpha' = \alpha + n\bar{X}_n, \quad \beta' = \beta + n \tag{3}$$

(As always, n is the sample size and \bar{X}_n is the sample mean.)

Solution: We have

$$f(x_1, \dots, x_n | \theta) = \frac{\theta^{\sum_{i=1}^n x_i} e^{-n\theta}}{\prod_{i=1}^n x_i!}$$

The prior distribution is

$$\pi(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\theta}$$

So the joint distribution of x_1, \dots, x_n and θ is

$$f(x_1, \dots, x_n, \theta) = \frac{\beta^\alpha}{\Gamma(\alpha) \prod_{i=1}^n x_i!} \theta^{\sum_{i=1}^n x_i + \alpha - 1} e^{-(\beta+n)\theta}$$

The posterior density is this function of θ normalized to make it a probability density (in θ). The above is proportional to

$$\theta^{\sum_{i=1}^n x_i + \alpha - 1} e^{-(\beta+n)\theta}$$

Without working out the normalization this shows that the posterior distribution is a gamma distribution with $\alpha' = \alpha + \sum_i x_i$ and $\beta' = \beta + n$.

(b) What is the Bayes estimator (using squared error loss) for θ ?

Solution: It is the mean of the posterior which is

$$\frac{\alpha'}{\beta'} = \frac{\alpha + \sum_{i=1}^n x_i}{\beta + n} = \frac{\alpha + n\bar{X}_n}{\beta + n}$$

(c) What is the limit as $n \rightarrow \infty$ of the Bayes estimator?

Solution: For large n the above is approximately

$$\frac{n\bar{X}_n}{n} = \bar{X}_n$$

So for large n the Bayes estimator is approximately the sample mean.

5. Suppose that a random sample is to be taken from a normal distribution with unknown mean θ and standard deviation is 2. The prior distribution of θ is normal with mean μ_0 and standard deviation 1.

(a) What is the smallest sample size that will insure the standard deviation of the posterior distribution of θ is at most 0.1 ?

(b) Suppose we use the squared error loss function. What is the smallest sample size that will insure that the risk is at most 0.01 ?

6. (566 students only) Let $c > 0$ and

$$L(\theta, a) = \begin{cases} c|\theta - a| & \text{if } \theta < a \\ |\theta - a| & \text{if } \theta \geq a \end{cases} \quad (4)$$

Assume that θ has a continuous distribution. Show that the Bayes estimator of θ is the $1/(1+c)$ quantile of the posterior distribution of θ .

Solution: The Bayes estimator is obtained by minimizing the average posterior loss:

$$\int_{-\infty}^{\infty} L(\theta, a) f(\theta|x_1, \dots, x_n) d\theta$$

With the above loss function, this is

$$c \int_{-\infty}^a (a - \theta) f(\theta|x_1, \dots, x_n) d\theta + \int_a^{\infty} (\theta - a) f(\theta|x_1, \dots, x_n) d\theta$$

Differentiating this with respect to a and setting the derivative to zero, we get an equation for the minimizing a :

$$0 = c(a - a) f(a|x_1, \dots, x_n) + c \int_{-\infty}^a f(\theta|x_1, \dots, x_n) d\theta - (a - a) f(a|x_1, \dots, x_n) - \int_a^{\infty} f(\theta|x_1, \dots, x_n) d\theta$$

which simplifies to

$$c \int_{-\infty}^a f(\theta|x_1, \dots, x_n) d\theta = \int_a^{\infty} f(\theta|x_1, \dots, x_n) d\theta$$

or

$$c \int_{-\infty}^a f(\theta|x_1, \dots, x_n) d\theta = 1 - \int_{-\infty}^a f(\theta|x_1, \dots, x_n) d\theta$$

or

$$\int_{-\infty}^a f(\theta|x_1, \dots, x_n) d\theta = \frac{1}{1+c}$$