Math 466/566 - Quiz 1 Solutions

1. A random variable X is uniformly distributed between 0 and 1 if the pdf is $f_X(x) = 1$ for $0 \le x \le 1$. The moment generating function of such a random variable is

$$M_X(t) = \frac{e^t - 1}{t} \tag{1}$$

You can use this fact without deriving it.

(a) Find the mean and variance of a random variable X that is uniformly distributed between 0 and 1.

Of course you can just do the integrals to compute these. To do it with the mgf, use Taylor expansion:

$$M_X(t) = \frac{t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots}{t} = 1 + \frac{1}{2}t + \frac{1}{6}t^2 + \cdots \tag{2}$$

Thus $E[X] = M_X'(0)$ is the coeffecient of t which is $1/2$. And $E[X^2] = M_X''(0)$ is 2 times the coeffcient of t^2 which is $1/3$. So $Var(X) = 1/3 - (1/2)^2 = 1/12$.

(b) Let $X_1, X_2, \cdots, X_n$ be independent random variables each of which is uniformly distributed between 0 and 1. Let

$$\bar{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{3}$$

be the usual sample mean. Find the mean and variance of $\bar{X_n}$.

The mean is just $1/2$. The variance is $\frac{1}{12n}$.

(c) For large n

$$P(0.5 \le \bar{X_n} \le 0.51) \approx \frac{1}{\sqrt{2\pi}} \int_0^{\alpha_n} e^{-z^2/2}\, dz \tag{4}$$

Find α_n. It should depend on n.

Let

$$Z_n = \frac{\bar{X_n} - 0.5}{1/\sqrt{12n}} \tag{5}$$

Then central limit thm says Z_n is approximately standard normal. And

$$P(0.5 \le \bar{X_n} \le 0.51) = P(0 \le Z_n \le \frac{0.51 - 0.5}{1/\sqrt{12n}}) = P(0 \le Z_n \le 0.01\sqrt{12n}) \tag{6}$$

Thus

$$\alpha_n = 0.01\sqrt{12n} \tag{7}$$

(d) 466: Find the moment generating function of $Y = \sum_{i=1}^{n} X_i$.
566: Find the moment generating function of $\bar{X}_n$.

The mgf of Y is the product of the mgf of the X_i. So

$$M_Y(t) = \left[\frac{e^t - 1}{t}\right]^n \tag{8}$$

To find the mgf of $\bar{X}_n$,

$$M_{\bar{X}_n}(t) = E[\exp(t\bar{X}_n)] = E[\exp(tY/n)] = M_Y(t/n) = \left[\frac{e^{t/n} - 1}{t/n}\right]^n \tag{9}$$