Math 466/566 - Quiz 4 - Solutions

1. Consider the following density

\[ f(x|\theta) = \theta^2 x e^{-\theta x}, \quad x \geq 0; \quad f(x|\theta) = 0, \quad x < 0 \]

It is easy to show the integral of this is 1, the mean is \( \mu = 2/\theta \) and the variance is \( \sigma^2 = 2/\theta^2 \).

(a) Find the maximum likelihood estimator \( \hat{\theta} \) of \( \theta \).

\[
\begin{align*}
    f(x_1, \cdots, x_n|\theta) &= \theta^{2n} \left( \prod_{i=1}^{n} x_i \right) \exp(-\theta \sum_{i=1}^{n} x_i) \\
    \ln(f(x_1, \cdots, x_n|\theta)) &= 2n \ln(\theta) + \sum_{i=1}^{n} \log(x_i) - \theta \sum_{i=1}^{n} x_i
\end{align*}
\]

Set the derivative equal to zero to find the maximum:

\[
\frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0
\]

So the MLE is

\[
\hat{\theta} = \frac{2n}{\sum_{i=1}^{n} x_i} = \frac{2}{\bar{X}_n}
\]

(b) Find the maximum likelihood estimator \( \hat{\mu} \) of the mean \( \mu \).

Since \( \mu = 2/\theta \), by functional invariance the MLE for \( \mu \) is

\[
\hat{\mu} = \frac{2}{\hat{\theta}} = \bar{X}_n
\]

Thus the MLE for \( \mu \) is just the sample mean.

(c) Suppose the sample size \( n \) is large. Describe the distribution of the estimators \( \hat{\theta} \) and \( \hat{\mu} \). (466 students had to just do one of them, 566 both.)

The MLE for \( \mu \) is the sample mean. By the central limit theorem, its distribution for large \( n \) is normal with mean \( \mu = 2/\theta \) and variance \( \sigma^2/n = 2/(n\theta^2) \).
For the MLE for $\theta$ we use theorem 8.5. It says the distribution of $\hat{\theta}$ is approximately normal with mean $\theta$ and variance equal to the Cramer-Rao bound $1/(nI(\theta))$. We compute $I(\theta)$:

$$I(\theta) = -\int \frac{\partial^2}{\partial \theta^2} \ln(f(x|\theta)) f(x|\theta) \, dx$$

$$= -\int \frac{\partial^2}{\partial \theta^2} (2 \ln(\theta) - \theta x) f(x|\theta) \, dx = \frac{2}{\theta^2}$$

Thus the variance is $\theta^2/(2n)$.

Instead of using the central limit theorem to find the distribution of $\hat{\mu}$, we can also use theorem 8.5. It says the distribution is approximately normal with mean approximately $2/\theta$ (in fact it is exactly equal to this) and variance approximately given by the Cramer Rao bound. To find the CR bound we need to write the density $f(x|\theta)$ in term of $\mu$.

$$f(x|\mu) = \frac{4}{\mu^2} x e^{-2x/\mu}$$

$$I(\mu) = -\int \frac{\partial^2}{\partial \mu^2} \ln(f(x|\mu)) f(x|\mu) \, dx$$

$$= -\int \frac{\partial^2}{\partial \mu^2} (-2\ln(\mu) - \frac{2x}{\mu}) f(x|\theta) \, dx$$

$$= \int \left( -\frac{2}{\mu^2} + \frac{4x}{\mu^3} \right) f(x|\theta) \, dx = \left( -\frac{2}{\mu^2} + \frac{4\mu}{\mu^3} \right) = \frac{2}{\mu^2}$$

(1)

So the CR bound is $\mu^2/(2n)$ which is the same as the variance we found using the central limit theorem.