Math 466/566 - Quiz 4 - Solutions

1. Consider the following density

$$f(x|\theta) = \theta^2\, x\, e^{-\theta x}, \quad x \geq 0; \qquad f(x|\theta) = 0, \quad x < 0$$

It is easy to show the integral of this is 1, the mean is $\mu = 2/\theta$ and the variance is $\sigma^2 = 2/\theta^2$.

(a) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

$$\begin{aligned} f(x_1, \cdots, x_n|\theta) &= \theta^{2n}\, (\prod_{i=1}^{n} x_i)\, \exp(-\theta \sum_{i=1}^{n} x_i) \\ ln(f(x_1, \cdots, x_n|\theta)) &= 2n \ln(\theta) + \sum_{i=1}^{n} log(x_i) - \theta \sum_{i=1}^{n} x_i \end{aligned}$$

Set the derivative equal to zero to find the maximum:

$$\frac{2n}{\hat{\theta}} - \sum_{i=1}^{n} x_i = 0$$

So the MLE is

$$\hat{\theta} = \frac{2n}{\sum_{i=1}^{n} x_i} = \frac{2}{\bar{X}_n}$$

(b) Find the maximum likelihood estimator $\hat{\mu}$ of the mean μ.

Since $\mu = 2/\theta$, by functional invariance the MLE for μ is

$$\hat{\mu} = \frac{2}{\hat{\theta}} = \hat{X}_n$$

Thus the MLE for μ is just the sample mean.
(c) Suppose the sample size n is large. Describe the distribution of the estimators $\hat{\theta}$ and $\hat{\mu}$. (466 students had to just do one of them, 566 both.)

The MLE for μ is the sample mean. By the central limit theorem, its distribution for large n is normal with mean $\mu = 2/\theta$ and variance $\sigma^2/n = 2/(n\theta^2)$.

For the MLE for θ we use theorem 8.5. It says the distribution of $\hat{\theta}$ is approximately normal with mean θ and variance equal to the Cramer-Rao bound $1/(nI(\theta))$. We compute $I(\theta)$:

$$\begin{aligned} I(\theta) &= -\int \frac{\partial^2}{\partial\theta^2} \ln(f(x|\theta))\, f(x|\theta)\, dx \\ &= -\int \frac{\partial^2}{\partial\theta^2} (2\ln(\theta) - \theta x)\, f(x|\theta)\, dx = \frac{2}{\theta^2} \end{aligned}$$

Thus the variance is $\theta^2/(2n)$.

Instead of using the central limit theorem to find the distribution of $\hat{\mu}$, we can also use theorem 8.5. It says the distribution is approximately normal with mean approximately $2/\theta$ (in fact it is exactly equal to this) and variance approximately given by the Cramer Rao bound. To find the CR bound we need to write the density $f(x|\theta)$ in term of μ.

$$f(x|\mu) = \frac{4}{\mu^2}\, x\, e^{-2x/\mu}$$

$$\begin{aligned} I(\mu) &= -\int \frac{\partial^2}{\partial\mu^2} \ln(f(x|\mu))\, f(x|\mu)\, dx \\ &= -\int \frac{\partial^2}{\partial\mu^2} (-2ln(\mu) - \frac{2x}{\mu})\, f(x|\theta)\, dx \\ &= \int (\frac{-2}{\mu^2} + \frac{4x}{\mu^3})\, f(x|\theta)\, dx = (\frac{-2}{\mu^2} + \frac{4\mu}{\mu^3}) = \frac{2}{\mu^2} \end{aligned} \tag{1}$$

So the CR bound is $\mu^2/(2n)$ which is the same as the variance we found using the central limit theorem.