1. The population has a uniform distribution on $[0, \theta]$ with $\theta$ unknown. So

$$f(x|\theta) = \frac{1}{\theta}1(0 \leq x \leq \theta)$$

We take a Bayesian point of view, and assume the prior distribution of $\theta$ is

$$\pi(\theta) = \begin{cases} 
3\theta^2 & \text{if } 0 \leq \theta \leq 1 \\
0 & \text{if } \theta \notin [0,1]
\end{cases}$$

We consider a random sample of size $n = 2$. 

(a) What is the density of the random sample given $\theta$, i.e., $f(x_1, x_2|\theta)$ ?

$$f(x_1, x_2|\theta) = \theta^{-2}1(0 \leq x_1, x_2 \leq \theta)$$

(b) What is the joint density of the random sample and $\theta$, i.e., $f(x_1, x_2, \theta)$ ?

$$f(x_1, x_2, \theta) = \pi(\theta)f(x_1, x_2|\theta) = 3 \cdot 1(0 \leq x_1, x_2 \leq \theta)1(0 \leq \theta \leq 1) = 3 \cdot 1(0 \leq x_1, x_2 \leq \theta \leq 1)$$

(c) Show that the posterior density of $\theta$, $\pi(\theta|x_1, x_2)$, is uniform on some interval and determine the interval. (It should depend on $x_1, x_2$.)

$$f(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1, x_2, \theta) \, d\theta = \int_{\max\{x_1, x_2\}}^{1} 3 \, d\theta = 3(1 - \max\{x_1, x_2\})$$

So the posterior is (for $x_1, x_2 \geq 0$)

$$\pi(\theta|x_1, x_2) = \frac{1(\max\{x_1, x_2\} \leq \theta \leq 1)}{1 - \max\{x_1, x_2\}}$$

Thus it is uniform on the interval $[\max\{x_1, x_2\}, 1]$.

(d) If we use squared error loss, what is the Bayes estimator of $\theta$ ?

The Bayes estimator is the mean of the posterior. The mean of a uniform distribution is just the average of the endpoints. So the Bayes estimator is

$$\frac{1 + \max\{x_1, x_2\}}{2}$$