Math 466/566 - Quiz 5

1. The population has a uniform distribution on $[0, \theta]$ with θ unknown. So

$$f(x|\theta) = \frac{1}{\theta} 1(0 \le x \le \theta)$$

We take a Bayesian point of view, and assume the prior distribution of θ is

$$\pi(\theta) = \begin{cases} 3\theta^2 & \text{if } 0 \le \theta \le 1 \\ 0 & \text{if } \theta \notin [0,1] \end{cases}$$

We consider a random sample of size $n = 2$.

(a) What is the density of the random sample given θ, i.e., $f(x_1, x_2|\theta)$?

$$f(x_1, x_2|\theta) = \theta^{-2}\, 1(0 \le x_1, x_2 \le \theta)$$

(b) What is the joint density of the random sample and θ, i.e., $f(x_1, x_2, \theta)$?

$$\begin{aligned} f(x_1, x_2, \theta) = \pi(\theta) f(x_1, x_2|\theta) &= 3 \cdot 1(0 \le x_1, x_2 \le \theta)\, 1(0 \le \theta \le 1) \\ &= 3 \cdot 1(0 \le x_1, x_2 \le \theta \le 1) \end{aligned}$$

(c) Show that the posterior density of θ, $\pi(\theta|x_1, x_2)$, is uniform on some interval and determine the interval. (It should depend on x_1, x_2.)

$$f(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1, x_2, \theta)\, d\theta = \int_{\max\{x_1, x_2\}}^{1} 3\, d\theta = 3(1 - \max\{x_1, x_2\})$$

So the posterior is (for $x_1, x_2 \ge 0$)

$$\pi(\theta|x_1, x_2) = \frac{1(\max\{x_1, x_2\} \le \theta \le 1)}{1 - \max\{x_1, x_2\}}$$

Thus it is uniform on the interval $[\max\{x_1, x_2\}, 1]$.

(d) If we use squared error loss, what is the Bayes estimator of θ ?

The Bayes estimator is the mean of the posterior. The mean of a uniform distribution is just the average of the endpoints. So the Bayes estimator is

$$\frac{1 + \max\{x_1, x_2\}}{2}$$