Math 466/566 - Quiz 6

1. The population has a uniform distribution on $[-\theta, \theta]$ with θ unknown. So

$$f(x|\theta) = \frac{1}{2\theta} 1(-\theta \leq x \leq \theta) = \frac{1}{2\theta} 1(0 \leq |x| \leq \theta)$$

We take a Bayesian point of view, and assume the prior distribution of θ is

$$\pi(\theta) = \begin{cases} 4\theta^3 & \text{if } 0 \leq \theta \leq 1 \\ 0 & \text{if } \theta \notin [0,1] \end{cases}$$

We consider a random sample of size $n = 3$.
(a) What is the density of the random sample given θ, i.e., $f(x_1, x_2, x_3|\theta)$?

Solution:

$$\begin{aligned} f(x_1, x_2, x_3|\theta) &= \frac{1}{8\theta^3} 1(0 \leq |x_i| \leq \theta, \; for\, i = 1, 2, 3) \\ &= \frac{1}{8\theta^3} 1(\max\{|x_i|\} \leq \theta) \end{aligned}$$

(b) Show that the posterior density of θ, $\pi(\theta|x_1, x_2, x_3)$, is uniform on some interval and determine the interval. (It should depend on x_1, x_2, x_3.)

Solution:

$$\begin{aligned} \pi(\theta|x_1, x_2, x_3) &= \frac{4\theta^3 1(0 \leq \theta \leq 1) \frac{1}{8\theta^3} 1(\max\{|x_i|\} \leq \theta)}{f(x_1, x_2, x_3)} \\ &= \frac{\frac{1}{2} 1(\max\{|x_i|\} \leq \theta \leq 1)}{f(x_1, x_2, x_3)} \qquad (1) \end{aligned}$$

We see that the only θ dependence is $1(\max\{|x_i|\} \leq \theta \leq 1)$ which shows that θ is uniformly distributed on $[\max\{|x_i|\}, 1]$. We don't need to work out $f(x_1, x_2, x_3)$. It is whatever is needed to normalize the above. So

$$\pi(\theta|x_1, x_2, x_3) = \frac{\frac{1}{2} 1(\max\{|x_i|\} \leq \theta \leq 1)}{1 - \max\{|x_i|\}} \qquad (2)$$

(c) If we use squared error loss, what is the Bayes estimator of θ ?

Solution: The mean of the posterior which is

$$\frac{1 + \max\{|x_i|\}}{2}$$