Math 520a - Homework 2

1. Use Cauchy’s integral formula (for an analytic function or its derivatives) to evaluate
   (a) For the contour \( \gamma(t) = e^{it}, \ 0 \leq t \leq 2\pi \), the integral
   \[
   \int_{\gamma} \frac{e^{iz}}{z^2} \, dz
   \]
   (b) For the contour \( \gamma(t) = 1 + \frac{1}{2}e^{it}, \ 0 \leq t \leq 2\pi \), the integral
   \[
   \int_{\gamma} \frac{\ln(z)}{(z-1)^n} \, dz
   \]

Solution: I’ll just give answers for this one.
(a) \(-2\pi\)
(b) Integral is 0 for \( n = 1 \). For \( n > 1 \) it is \((-1)^n2\pi i/(n-1)\).

2. Let \( f(z) \) be an entire function such that there are constants \( C, D \) with
   \[
   |f(z)| \leq C + D|z|^n, \ \forall z
   \]
   Prove that \( f \) is a polynomial of degree at most \( n \).

Solution: Since \( f \) is entire it has a power series about the origin which converges for all \( z \).

\[
 f(z) = \sum_{k=0}^{\infty} a_k z^k
\]

The coefficients are given by \( a_k = f^{(k)}(0)/k! \). We will show that \( f^{(k)}(0) = 0 \) for \( k > n \). This implies \( a_k = 0 \) for \( k > n \) and so the power series is just a polynomial.

By considering a circle of radius \( R \), Cauchy’s inequality says

\[
|f^{(k)}(0)| \leq \frac{k!M_R}{R^k}
\]

where \( M_R \) is the sup of \( |f(z)| \) over the circle of radius \( R \). By the hypothesis, \( M_R \leq C + DR^n \). For \( k > n \), \( (C + DR^n)/R^k \to 0 \) as \( R \to \infty \), and so \( f^{(k)}(0) = 0 \)
3. Let \( \Omega \) be a region (connected open set). Suppose that \( f \) and \( g \) are analytic functions on \( \Omega \) such that \( f(z)g(z) = 0 \) for all \( z \in \Omega \). Prove that at least one of \( f \) and \( g \) is identically zero on \( \Omega \).

**Solution:** We can find a point \( z_0 \in \Omega \) and a sequence \( z_n \in \Omega \) which converges to \( z_0 \) but never equals \( z_0 \). For every \( n \), \( f(z_n)g(z_n) = 0 \), and so either \( f(z_n) = 0 \) or \( g(z_n) = 0 \). So one of the sets \( \{ n : f(z_n) = 0 \} \) and \( \{ n : g(z_n) = 0 \} \) must be infinite. Assume the first one is infinite. Then there is a subsequence \( z_{n_k} \) with \( f(z_{n_k}) = 0 \). But this implies \( f \) is identically 0 on \( \Omega \).

4. Let \( f \) be entire and suppose that for every \( z_0 \), the power series expansion about \( z_0 \)

\[
  f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n
\]

has at least one coefficient \( a_n \) which is zero. (Note that the \( a_n \) depend on \( z_0 \).) Prove that \( f \) is a polynomial. This is problem 13 on p. 67 in the book. You can find a hint there.

**Solution:** Note that for the power series about \( z_0 \), \( a_n = 0 \) is equivalent to \( f^{(n)}(z_0) = 0 \). So the hypothesis says that for every \( z_0 \), there is an \( n \) for which \( f^{(n)}(z_0) = 0 \). Now let \( A_n = \{ z : |z| \leq 1, f^{(n)}(z_0) = 0 \} \). The union of the \( A_n \) is the unit disc and so is uncountable. So at least one \( A_n \) is uncountable (and hence infinite). Let \( m \) be such that \( A_m \) is infinite. Then there is a sequence \( z_l \) of distinct elements in \( A_m \). Since the closed unit disc is compact, it has a convergent subsequence. Since \( f^{(m)} \) vanishes on this subsequence, \( f^{(m)} \) is identically zero. So \( f \) is a polynomial.

5. Let \( D \) be an open disc. Suppose that \( f \) is continuous on \( \overline{D} \), analytic on \( D \) and that \( f \) never vanishes on \( \overline{D} \). Suppose also that \( |z| = 1 \Rightarrow |f(z)| = 1 \). Prove that \( f \) is constant. This is problem 15 on p. 67 in the book. You can find a hint there.

**Solution:** Discussed in class.

6. Let \( g(t) \) be continuous on \([0, \infty)\) with \( \int_0^\infty |g(t)| \, dt < \infty \). Define

\[
  f(z) = \int_0^\infty \cos(z + t) \, g(t) \, dt
\]

Prove that \( f(z) \) is entire. For complex \( z \), \( \cos(z) \) is defined to be \( (e^{iz} + e^{-iz})/2 \). (Caution: for complex \( z \) we do not have \( |\cos(z)| \leq 1 \).)
Solution: Define

$$f_n(z) = \int_0^n \cos(z + t) g(t) \, dt$$

For a fixed $t$, $z \to \cos(z + t) g(t)$ is entire. Also, $\cos(z + t) g(t)$ is jointly continuous in $t$ and $z$. By the theorem proved in class, $f_n$ is entire. We will prove it converges uniformly on compact subsets of the plane to $f$. This will prove $f$ is analytic.

Every compact subset of the plane is contained in the strip $|\text{Im}(z)| \leq M$ for some $M > 0$. So it suffices to prove uniform convergence on such a strip.

$$|f(z) - f_n(z)| = \left| \int_n^\infty \cos(z + t) g(t) \, dt \right|$$

For $z = z + iy$, on the strip we have

$$|\cos(z + t)| = \frac{1}{2} |e^{iz} + e^{-iz}| \leq \frac{1}{2} (|e^z| + |e^{-iz}|) = \frac{1}{2} (|e^{-y}| + |e^y|) \leq e^M$$

Hence

$$|f(z) - f_n(z)| \leq e^M \int_n^\infty |g(t)| \, dt$$

This bound holds for all $z$ in the strip and the right side is independent of $z$ and goes to 0 as $n \to \infty$ proving the needed uniform convergence.

7. Let $\Omega$ be open. Let $f_n, f$ be analytic on $\Omega$ and suppose that for all circles $C$ such that the circle and its interior are in $\Omega$, $f_n$ converges uniformly to $f$ on $C$. Prove that $f_n$ converges uniformly to $f$ on all compact subsets of $\Omega$.

Solution: Let $K$ be a compact subset of $\Omega$. For each $z \in K$ we can find $\epsilon_z > 0$ such that $B_{3\epsilon_z}(z) \subset \Omega$. (Note the factor of $3$.) The discs $B_{\epsilon_z}(z)$ as $z$ ranges over $K$ are an open cover of $K$. (Note there is not a factor of $3$ here.) So there is a finite subcover, i.e., there are $z_1, \ldots, z_n \in K$ such that

$$K \subset \bigcup_{j=1}^n B_{\epsilon_{z_j}}(z_j)$$

Since there are finite number of discs in the cover, it suffices to show the convergence is uniform on each disc. To simplify the notation, let $B_{\epsilon}(\zeta)$ be one of the discs.
We know $B_{3\epsilon}(\zeta) \subset \Omega$. Let $C$ be the circle centered at $\zeta$ with radius $2\epsilon$. Then for $z \in B_{2\epsilon}(\zeta)$, we have

$$f(z) - f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(w) - f_n(w)}{w - z} \, dw$$

If $z \in B_{\epsilon}(\zeta)$, we have $|w - z| \geq \epsilon$ for $w \in C$ and so

$$|f(z) - f_n(z)| \leq \frac{1}{2\pi} \frac{1}{\epsilon} |C| \|f - f_n\|_C$$

where $\|f - f_n\|_C$ is the sup of $|f(w) - f_n(w)|$ over $w \in C$ and $|C|$ is the length of $C$ which is just $4\pi\epsilon$. So

$$|f(z) - f_n(z)| \leq 2\|f - f_n\|_C$$

Note that the right side is now independent of $z$ and goes to zero as $n \to \infty$ since $f_n$ converges uniformly to $f$ on $C$. This completes the proof.