Math 520b - Homework 2

1. Let $H$ be a Hilbert space. Let $E_1, E_2$ be closed subspaces which are orthogonal. Prove there is a third closed subspace $E_3$ which is orthogonal to both $E_1$ and $E_2$ and such that every $v \in H$ can be written in a unique way as $v = v_1 + v_2 + v_3$ where $v_i \in E_i$. Show that $E_3 = (E_1 \oplus E_2)^\perp = E_1^\perp \cap E_2^\perp$. You may assume the projection theorem as stated in the notes or in Farkas and Kra.

2. (From Farkas and Kra) (a) Let $f \in L^2[0,1]$. Show that $f$ is equal to a constant almost everywhere if and only if
   \[ \int_0^1 f(z)g'(x)dx = 0 \]  
   (1)
   for all $C^\infty$ functions with compact support. You can think of this as a 1d analog of Weyl’s lemma.
   (b) If we place the hypothesis (1) by
   \[ \int_0^1 f(z)g''(x)dx = 0 \]  
   (2)
   what can you conclude about $f$?

3. Let $\phi$ be a $C^2$ function on the unit disc that is harmonic. Prove that $\phi$ is $C^\infty$.

4. Prove parts (a) and (b) of the corollary to theorem 2 which says that $H$ is the set of harmonic forms.

5. A meromorphic differential (or 1-form) $\omega$ is a 1-form such that in every chart $z$, $\omega = f(z)dz$ where $f(z)$ is meromorphic. The form is said to have a pole at $p$ if $f$ does. Suppose $\omega$ has a pole at $p$. Let $z$ be a chart centered at $p$. Writing $\omega = f(z)dz$, $f(z)$ has a Laurent series:
   \[ f(z) = \sum_{n=N}^{\infty} a_n z^n \]
   where $N$ is negative. We define the residue of the form at $p$ to be $a_{-1}$. In general the coefficients in the Laurent series depend on the choice of chart. Prove that the residue does not depend on the choice of chart. Hint: how do you compute the coeff in a Laurent series?