Math 520b - Homework 4

1. Let $M$ be a compact Riemann surface of genus $g$. Let $P_1, \ldots, P_k \in M$ and let $n_1, \ldots, n_k$ be positive integers. Find the dimension of the vector space of meromorphic forms whose poles are a subset of $\{P_1, \ldots, P_k\}$ and such that the order of the pole at $P_j$ (if there is one) is at most $n_j$.

2. Prove that for the Riemann sphere two divisors $U_1$ and $U_2$ are equivalent if and only if $\deg(U_1) = \deg(U_2)$.

3. Let $\tau \in \mathbb{C}$ have positive imaginary part. Let $L$ be the lattice $\mathbb{Z} + \mathbb{Z} \tau$. Let $M$ be the Riemann surface which is the torus $\mathbb{C}/L$. The meromorphic functions on $M$ are the meromorphic functions on $\mathbb{C}$ which are doubly periodic with periods 1 and $\tau$. For $p \in \mathbb{C}$ let $\gamma_p$ be the closed contour which is the parallelogram with vertices $p, p + 1, p + 1 + \tau, p + \tau$ traversed in that order. Let $f$ be a doubly periodic meromorphic function on $\mathbb{C}$. Show that for $p$ such that $\gamma_p$ does not contain any poles or zeroes of $f$, we have

$$\frac{1}{2\pi i} \int_{\gamma_p} \frac{h'(z)}{h(z)} dz \in L$$

4. Let $M$ be the torus defined in the previous problem. Consider the divisor $U = P_1^{n_1} \cdots P_k^{n_k}$. We know that if it is principal, then $\deg(U) = 0$. Define

$$A(U) = \sum_{j=1}^{k} n_j P_j$$

(Addition of points in the torus $M$ is addition in $\mathbb{C}$ mod $L$.) Prove that if $U$ is principal, then $A(U) = 0$. (We think of $A(U)$ as an element of the torus, so $A(U) = 0$ means it is $0$ mod $L$.) This is “half” of Abel’s theorem for the torus which says that a divisor $U$ is principal if and only if $\deg(U) = 0$ and $A(U) = 0$.

5. Let $M$ be a compact Riemann surface. Let $P_1, P_2, \ldots, P_n$ be distinct points in $M$. Prove that there is a meromorphic form such that for all $j$ the form is non-zero at $P_j$ and does not have a pole at $P_j$. Hint: Let $Q$ be a point distinct from all the $P_j$. For large $N$ what is the dimension of $L(Q^{-N})$ and of $L(Q^{-N}P_j)$?