Math 520b - Homework 6

1. (Mirandi, p. 112 G) Let $h(x)$ be a polynomial of degree $2g + 1$ or $2g + 2$ with distinct roots. Let $M$ be the hyperelliptic Riemann surface given by $y^2 = h(x)$. Show that $dx/y$ is a holomorphic 1-form on $M$. More generally, show that if $p(x)$ is a polynomial of degree at most $g - 1$, then $p(x)dx/y$ is a holomorphic 1-form. Prove that every holomorphic 1-form is of this form.

2. (Mirandi, p. 137 A) Let $M$ be the hyperelliptic surface given by $y^2 = x^5 - x$. Find the principal divisors $(x)$ and $(y)$.

3. (Mirandi, p. 84 H) Let $\hat{C}$ be the Riemann sphere and let $z$ be the chart which comes from stereographic projection with respect to the north pole. Let $R_n$ be the conformal automorphism $R_n(z) = \exp(2\pi i/n)z$ (rotation by $2\pi/n$). Let $F$ be the conformal automorphism $F(z) = 1/z$. Let $H$ be the group generated by $R_n$ and $F$. What is this group? Find the branch points and their ramification number for the quotient map $\pi : \hat{C} \to \hat{C}/H$.

4. Let $H$ be a finite group of conformal automorphisms of a Riemann surface. Recall that for $P \in M$, $H_P$ is the stabilizer subgroup of $P$, i.e., the set of $h \in H$ such that $h(P) = P$. Show that the points $P$ for which $H_P$ is not trivial are isolated, i.e., if $P$ is such a point then there is a neighborhood of $P$ in which $P$ is the only point with a nontrivial stabilizer subgroup.

5. Prove Harnack’s inequality which I stated in class:

Let $D$ be a domain in $\mathbb{C}$, $D_1$ a bounded subdomain of $D$ such that $\delta = \text{dist}(D_1, \partial D) > 0$. Then there is a constant $c = c(D_1, D)$ such that for all positive harmonic functions $u$,

$$\frac{1}{c} \leq \frac{u(z_1)}{u(z_2)} \leq c, \quad \forall z_1, z_2 \in D_1$$

Note that the constant $c$ does not depend on $u$. 

1