1 Introduction

The main goal of this course is to understand SLE and its exciting connections to a variety of discrete models from probability and statistical physics. SLE is a one parameter family of stochastic processes that produce random curves in the plane. It was introduced by Oded Schramm in a paper that appeared in 1999. He named it Stochastic Loewner Evolution, but some authors now refer to it as Schramm-Loewner Evolution.

To understand SLE and its importance requires understanding various areas of mathematics and physics. So in addition to the SLE theme, the course will have several secondary themes. On the physics side we will learn about several models that are defined on lattices in the plane and their *scaling or continuum limits.* We are primarily interested in these models when they are at a *critical point.* Critical points are values of the parameters of the system with the property that the randomness of the system, which typically is seen only at microscopic length scales, produces random structures that can be seen at macroscopic length scales. In the scaling limit critical phenemona are especially interesting because they exhibit *universality.* Universality means that many different models have the same scaling limit. Thus a very important problem is to determine what are the possible scaling limits.

In two dimensions, many models from statistical physics are *conformally invariant.* Roughly speaking conformal invariance means that the model is not changed by a map in the plane that preserves angles. If we think of the plane as the complex plane, then conformal maps are analytic functions. There are lots of such maps of the plane, and the rich nature of this symmetry produced in the 1980's a field now known as conformal field theory which revolutionized our understanding of two-dimensional critical phenomena. The results of the past few years on SLE are advancing our understanding of the geometry of two-dimensional critical phenomena in an equally exciting fashion. One of our subthemes on the mathematics side will be conformal maps.

The equation that defines SLE is a *stochastic differential equation.* Techniques from stochastic differential equations are used frequently in the study of SLE and its connection to statistical physics models. So another subtheme will be stochastic differential equations.

Critical phenomena

Many physical systems can undergo phase transitions. Roughly speaking a phase transition means that a tiny change in one of the parameters, e.g., temperature, can produce a not so tiny change in some physical quantity, e.g., magnetization, density, susceptibility, ... The behavior at a phase transition is often called critical behavior. Mathematically, phase transitions can be characterized as follows. For most values of the parameters like temperature, density, ... the physical quantities are analytic functions of the parameters. Critical phenomena occur at points (or curves, surfaces...) in the parameter space where some physical quantity is not analytic in the parameters.

The above is the standard physics description of a phase transition. There is a more probabilistic definition of a phase transition that is probably more relevant to the phenomena we will study in this course. The models from statistical physics and probability that we will study are macroscopic models that are made up of a huge number of microscopic components which have some randomness. One wants to study these models in the limit that the number of microscopic components goes to infinity. In this limit, one typically shrinks the lattice on which the model is defined so that in the limit one has some continuum model. Away from a critical point, this scaling limit is deterministic. For example, if the discrete model is a model of random curves, in the scaling limit the curve will be a non-random, e.g., just a straight line or perhaps a point. In a model with interfaces, these interfaces will be flat and non-random in the scaling limit So away from the critical point one has law of large number type results. The critical point is characterized by the scaling limit exhibiting randomness. For discrete models of random curves, the scaling limit will have random curves. For models with interfaces, the scaling limit will have random interfaces. These random curves and interfaces typically have a fractal nature.

A quick look at SLE

We will give a very quick overview of the definition of SLE. We work in the upper half of the complex plane which we denote by $\mathbf{H}$. Suppose that $\gamma(t)$ is a curve that starts at the origin and stays in the upper half plane. We let $\gamma[0,t]$ denote the image of the curve up to time t. Then for each t, $\mathbf{H} \setminus \gamma[0,t]$ is a simply connected set. By the Riemann mapping theorem, there is a conformal map g_t from $\mathbf{H} \setminus \gamma[0,t]$ onto $\mathbf{H}$. We can normalize it by $g_t(0) = 0$, $g_t(\infty) = \infty$, $g_t'(\infty) = 1$. Of course, g_0 is just the identity, $g_0(z) = z$. Loewner showed that $g_t(z)$ satisfies the following differential equation,

$$\partial_t g(z) = \frac{2}{g_t(z) - \sqrt{\kappa} U_t} \tag{1}$$

where U_t is a real-valued function known as the driving function. It is given by $U_t = g_t(\gamma(t))$, i.e., U_t is the image of the tip of the curve. One can try to go the other way, take a diving function U_t, define g_t by the above and then try to recover the curve γ. It turns out that for continuous U_t, g_t need not come from a curve γ.

SLE is obtained by taking the driving function to be Brownian motion times a parameter:

$$\partial_t g(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t} \tag{2}$$

B_t is a standard Brownian motion and κ is the parameter. Now g_t is a random conformal map. It maps a simply connected subset of the upper half plane onto the upper half plane. We can write this simply connected set as $\mathbf{H} \setminus K_t$ where K_t is a compact subset of the upper half plane. The set K_t is a random set which grows with t. As we will see, if $\kappa < 4$ then K_t is a curve, but for larger κ it is not. Nonetheless, for $\kappa > 4$ it is possible to define a curve which generates K_t in some sense.

Let

$$\hat{g}_t(z) = \frac{g_t(z) - \sqrt{\kappa} B_t}{\sqrt{\kappa}} \tag{3}$$

Then $\hat{g}_t(z)$ satisfies the stochastic differential equation.

$$d\hat{g}_t(z) = \frac{2}{\kappa \hat{g}_t(z)} dt + dW_t \tag{4}$$

with $\hat{g}_0(z) = z$. Here $W_t = -B_t$, so W_t is another standard Brownian motion.

Connections with discrete models

We give a very quick list of the discrete models that SLE is believed to describe. In some cases the connection is now a theorem, in others it is only a conjecture. We will not define any of these models here. The reader who has not seen them before should not panic. We will devote a chapter to defining and explaining these models.

The *loop-erased random walk (LERW)* is formed by taking an ordinary random walk and erasing the loops it forms as we follow the walk. Schramm, Lawler and Werner have proved that the scaling limit of the LERW is SLE_2.

The LERW gives a measure on the set of random walks that do not intersect themselves. There is a different measure on the set of these walks, namely the uniform measure. This model is known as the *self-avoiding walk.* It is conjectured to converge to $\mathrm{SLE}_{8/3}$ in the scaling limit. There is numerical support for this conjecture, but no proof.

A two-dimensional Brownian motion traces a curve in the plane that intersects itself many times. The outer boundary or *frontier* of this set is a fractal object. Mandelbrot conjectured that the Hausdorff dimension of this frontier is 4/3. This was proved using SLE techniques by Lawler, Schramm and Werner. This is the same fractal dimension as the conjectered dimension of the SAW and in fact the frontier of 2d Brownian motion is in some sense the same as an $\mathrm{SLE}_{8/3}$ curve.

Consider a tiling of the plane by hexagons. For each hexagon flip a fair coin and color the hexagon white for heads, black for tails. This defines a *critical percolation.* Do this in a large square and fix the hexagons to be white around half of the boundary, black around the other half. This creates an interface which has black hexagons on one side and white hexagons on the other side. This interface has been proved to converges to SLE_6.

The interface in the above percolation model can be defined by an exploration process in which one generates the interface by following the interface and flipping the coin to determine the color of the hexagonal only when we encounter the hexagon. The harmonic explorer is a similar exploration process in which the random rule for determining the color of the hexagon is more complicated. (It involves discrete harmonic functions.) Schramm and Sheffield have proved that its scaling limit is SLE_4.

Consider a finite piece of a two-dimensional lattice. A spanning tree is a subset of the bonds in the lattice that connects all the sites but has no loops. The *uniform spanning* *tree* is the uniform measure on the set of all such spanning trees. Its scaling limit is conjectured to be related to SLE_8.

In the *Ising model* there is a random variable at each site of the lattice, called the spin, which can be ± 1. The Gibbs measure gives a probability measure on the space of all spin configurations. It depends on a parameter, the temperature, and there is one value of the temperature where the system is critical. One can force an interface in the system by appropriate boundary conditions. It is conjectured that this interface is related to $SLE_16/3$?. The *q-state Potts model* is similar to the Ising model but with q possible values for the spin at each site. It is conjectured to be related to SLE_κ where κ is related to q by

$$q = 2 + 2\cos(8\pi/\kappa) \tag{5}$$

with $4 \leq \kappa \leq 8$.

Because of universality one might hope to determine all the possible scaling limits of a statistical mechanical system. This problem is of greatest interest in three dimensions since most physical systems (but not all) are three dimensional. In three dimensions this is probably an impossible problem. In two dimensions, conformal invariance makes tremendous progress possible. What is exciting about SLE is that it gives a one parameter family of possible scaling limits, and it appears that the scaling limits of many two-dimensional models are given by SLE for some value of the parameter.

Unfortunately, SLE is applicable only to two-dimensional models. In three dimensions conformal invariance is not of any use in determining the possible scaling limits. Another criticism of SLE is that for some models it only give a partial description. For example, in the Ising model SLE is believed to describe interfaces, but there are other critical aspects of the model.

A very simple example

To begin to understand the ideas of scaling limits and critical phenomena, we will consider a very simple example - the scaling limit of a one-dimensional random walk. We take a coin which has probability p of heads, $1-p$ of tails. We flip it infinitely many times. Let $X_i = +1$ when the ith flip is heads, $= -1$ when the ith flip is tails. Let

$$S_n = \sum_{i=1}^{n} X_i \tag{6}$$

We graph S_n as a function of n.

For unfair coin the scaling limit is a straight line. It is not random, so this is not a critical system.

For fair coin, the scaling is different. The graph converges to a complicated (fractal) curve which is random. If repeat the coin flips, the graph and the scaling limit we get a completely different curve. Universality in this model means the following. The X_i is an i.i.d. sequence and we can carry out the above with some other i.i.d. sequence. If the mean of X_i is zero and its variance is finite, then we will get essentially the same scaling limit. This is a well known theorem from mathematics.

We conclude the introduction with a brief summary of the chapters to come.

2. Brownian motion: As we saw in our quick look at SLE, Brownian motion plays a key role in SLE. It is also of interest as one of the models SLE helps describe. This chapter will define Brownian motion and state (without proofs) its important properties.

3. Statistical physics models: We will define the various discrete models from statistical physics and probability that we mentioned above, and describe some of their behavior.

4. Conformal maps: We will define conformal maps and state the Riemann mapping theorem. Some of the properties of these maps that are needed for SLE will be proved, in particular the Loewner differential equation we saw above. Finally, we will will discuss what conformal invariance means for the scaling limits of the models we introduced in the previous chapter.

5. SLE: We will define SLE and state a few of its key properties. We will make precise the connections with the scaling limits of the various discrete models. This will be a short chapter. Proving anything about SLE requires the stochastic calculus.

6. Stochastic differential equations: This will be a crash course in SDE's following the chapter in Lawler's book.

7. Properties of SLE As the parameter κ varies SLE has three distinct phases. We will prove this. We will also define the restriction properties and the locality properties and prove that SLE satisfies these properties when $\kappa = 6$ and $8/3$, respectively.

8. Convergence of discrete models to SLE: If there is any time left we will sketch one of the proofs that SLE gives the scaling limit of a discrete model. Most likely, we will look at the proof that the loop-erased random walk converges to SLE_2.