4 Conformal maps

4.1 Definition, Riemann mapping theorem

We start by restating a definition we made before.

Definition 13 *Let D and D' be open subsets of $\mathbb{R}^2$. A map $f : D \to D'$ is said to preserve angles if for every two differentiable curves γ_1 and γ_2 in D defined on the time interval $(-\epsilon, \epsilon)$ which intersect at $t = 0$, the angle formed by their tangents at $\gamma_i(0)$ is equal to the angle formed by the tangents to $f \circ \gamma$ and $f \circ \gamma'$ at $f(\gamma_i(0))$. A conformal map from D to D' is a one to one, onto, differentiable function that preserves angles.*

Let $f(x, y) = (u(x, y), v(x, y))$ be a differentiable map. We leave it to the reader to show that it preserves angles at a point if and only if its derivative (which is a 2 by 2 matrix) is equal to a positive constant times a rotation matrix. So there is an $a > 0$ and a θ such that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = a \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{67}$$

This implies that the Cauchy-Riemann equations are satisfied. Conversely, if the Cauchy-Riemann equations are satisfied and the derivative at the point is not zero, then one can show there is an $a > 0$ and θ such that the above is true. So the map preserves angles. Thus a map is a conformal map if and only if it is a one to one, onto analytic function of D to D'. Note if f is a conformal map of D onto D', then f^{-1} is a conformal map of D' onto D.

There is a special family of conformal maps - the linear fractional transformations. They are of the form

$$f(z) = \frac{az + b}{cz + d} \tag{68}$$

where a, b, c, d are complex numbers with $ad - bc \neq 0$. Linear fractional transformations map circles onto circles if we think of lines in the plane as circles.

We will use $\mathbb{D}$ to denote the open unit disc with center at the origin and $\mathbb{H}$ to denote the upper half plane. The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is the complex plane with the point at ∞ added and the usual topology. Linear fractional transformations are conformal maps of $\hat{\mathbb{C}}$ onto itself. The linear fractional transformations that map $\mathbb{D}$ onto $\mathbb{D}$ are of the form

$$f(z) = e^{ia} \frac{z - w}{1 - \overline{w} z}, \quad w \in \mathbb{D}, \quad a \in \mathbb{R} \tag{69}$$

The extended half plane $\hat{\mathbb{H}}$ is $\mathbb{H}$ with ∞ added. The linear fractional transformations that map $\hat{\mathbb{H}}$ onto $\hat{\mathbb{H}}$ are of the form

$$f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d, \in \mathbb{R}, \quad ad - bc > 0 \tag{70}$$

a conformal map of the unit disc $\mathbb{D}$ to the upper half plane $\mathbb{H}$ is

$$f(z) = \frac{z-i}{z+i} \tag{71}$$

All the conformal maps of $\mathbb{D}$ onto $\mathbb{H}$ are obtained by following this map with a conformal map of $\mathbb{H}$ onto itself.

Theorem 14 *(Schwarz lemma) If $f : \mathbb{D} \to \mathbb{D}$ is analytic with $f(0) = 0$ then $|f(z)| \le |z|$ for $z \in \mathbb{D}$. If equality holds for a nonzero z, then $f(z) = e^{i\theta} z$ for some real θ.*

Proof: Apply the maximum principle to $f(z)/z$.

We will use "domain" to mean an open subset of the complex plane. A domain is simply connected if it does not have any holes:

Definition 14 *A domain D is simply connected if the region bounded by every simple closed curve in D is contained in D, i.e., every simple closed curve in D may be continuously contracted to a point without leaving D. Equivalently, D is simply connected if $\hat{\mathbb{C}} \setminus D$ is connected.*

Recall that Cauchy's theorem says that if D is simply connected and f is analytic on D and γ is a differentiable closed curve in D, then

$$\int_\gamma f(z) dz = 0 \tag{72}$$

We can take the logarithm of a non-zero analytic function on a simply connected domain.

Proposition 5 *Let D be a simply connected domain, f an analytic function on D which never vanishes. Then there exists an analytic function g on D such that $f = e^g$.*

To see that the simply connected hypothesis is needed, consider $f(z) = z$ on the plane with the origin removed. There is no way to define $\log(z)$ on this region. You must have a branch cut somewhere.

We will use the following theorem on a daily basis.

Theorem 15 *(Riemann mapping theorem) Let D be a simply connected region which is not all of $\mathbb{C}$ and let $w \in D$. Then there is a unique conformal transformation f of D onto the unit disc $\mathbb{D}$ such that $f(w) = 0$ and $f'(w) = 0$.*

Corollary: Any two simply connected domains have a conformal map between them.

A proof can be found in Lawler's book or any first year graduate complex variables book. We do not give it in full, but there is a consequence of the proof we will need later so we give the main idea. The proof considers the collection of conformal maps h of D to $\mathbb{D}$ with $h(w) = 0$ and $h'(w) > 0$ which are not necessarily onto. The proof shows that there is an f in this collection which maximizes $h'(w)$. One then shows that this f is onto. Thus for any conformal map h from D into $\mathbb{D}$ with $h(w) = 0$ and $h'(w) > 0$ we have $h'(w) \leq f'(w)$.

One of the amazing aspects of this theorem is that it does not require any smoothness of the boundary. The boundary need not even be a curve. If we want to extend the conformal map so that it maps the boundary of D onto the boundary of $\mathbb{D}$ then we need some condition on the boundary of D

Definition 15 *A closed set $K \subset \mathbb{C}$ is locally connected if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $z, w \in K$ with $|z - w| < \delta$ then there exists a connected set $K_1 \subset K$ with $z, w \in K_1$ and $diam(K_1) < \epsilon$.*

Theorem 16 *Let D be a bounded simply connected domain. Let f be a conformal map of the unit disc $\mathbb{D}$ onto D. Then f has a continuous extension to $\overline{\mathbb{D}}$ if and only if $\mathbb{C} \setminus D$ is locally connected.*

We will not give a proof. A reference for a proof may be found in Lawler's book. The boundary of a simply connected domain need not be the image of a curve. If it is the image of a continuous curve, then ∂D is locally connected. With a little more work one can then show that $\mathbb{C} \setminus D$ is locally connected. Conversely, if $\mathbb{C} \setminus D$ is locally connected, then by the above theorem and the Riemann mapping theorem there is a conformal map of $\mathbb{D}$ onto D which extends continuously to the boundary of D. Consider the curve $f(e^{it})$ where $0 \leq t \leq 2\pi$. It will traverse the boundary of D. Thus the boundary is a curve if and only if $\mathbb{C} \setminus D$ is locally connected.

Consider the slit disc $D = \mathbb{D} \setminus [0, 1)$ where $[0, 1)$ denotes the subset of the complex plane consisting of real numbers in the interval. This is a simply connected domain and its boundary is a curve. However, the points along the slit are traversed twice when we traverse the curve that gives the boundary. A closed curve $\gamma : [a, b] \to \mathbb{C}$ is a Jordan curve if γ is one to one on $[a, b)$. A bounded domain is a Jordan domain if its boundary is a simple curve. For a Jordan domain D, if f is a conformal map of $\mathbb{D}$ onto D, then the continuous extension of f to $\overline{\mathbb{D}}$ will be one to one.

Proposition 6 *Let D,D' be Jordan domains.*
(i) Let $z_1, z_2, z_3 \in \partial D$ and $z'_1, z'_2, z'_3 \in \partial D'$. Both sets of points are oriented clockwise. Then there is a unique conformal map f of D onto D' such that $f(z_i) = z'_i, i = 1, 2, 3$.
(ii) Let $z \in \partial D$, $w \in D$ and $z' \in \partial D'$, $w' \in D'$. Then there is a unique conformal map f of D onto D' such that $f(z) = z'$ and $f(w) = w'$.

Proof: Let f be a conformal map of D onto $\mathbb{D}$ and g a conformal map of D' onto $\mathbb{D}$. So $f(z_i)$ and $g(z_i')$ belong to $\partial\mathbb{D}$. If h is a conformal map of $\mathbb{D}$ onto itself such that $h(f(z_i)) = g(z_i')$ for $i = 1, 2, 3$, then $g^{-1} \circ h \circ f$ is the desired map of D to D'. The conformal maps of $\mathbb{D}$ onto itself are linear fractional transformations of the form (69). We leave it as an exercise to show that a suitable choice of a and θ gives the needed h.

Roughly speaking, the family of conformal maps from one simply connected domain to another has three real degrees of freedom. In (i) they are determined by three real constraints. In (ii) the constraint $f(w) = w'$ is a complex constraint and so uses two real degrees of freedom. Another common way to impose a real constraint is to require that the derivative at some interior point be positive as we did in the Riemann mapping theorem.

We end this section with a useful result to extending analytic functions. If $f(z)$ is analytic, then it is easy to check that $\overline{f(\overline{z})}$ is too (defined on the obvious domain). We would like to use this fact to take an analytic function defined on a subset of the upper half plane and extend it to the refected domain in the lower half plane. Clearly $f(z)$ will need to be real on the part of the real axis in the original domain if this is to work.

Theorem 17 *(Schwarz reflection principle) Let D be a domain which is symmetric about the real axis. Let $D_+ = D \cap \mathbb{H}$. Let f be a function which is continuous on $\overline{D}_+$, analytic in D_+ and real valued on the set of reals in D. Then f may be extended to an analytic function on all of D which satisfies $f(\overline{z}) = \overline{f(z)}$.*

4.2 Univalent functions

A function is called univalent if it is analytic and one to one. Let $\mathcal{A}$ be the set of simply connected domains not equal to the entire plane which contain the origin. Let $\mathcal{S}^*$ denote the set of univalent functions on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$. The Riemann mapping theorem says that $f \to f(\mathbb{D})$ gives a one to one correspondence between $\mathcal{S}^*$ and $\mathcal{A}$. We let $\mathcal{S}$ denote the set of $f \in \mathcal{S}^*$ with $f'(0) = 1$.

A *compact hull* K is a connected compact subset of $\mathbb{C}$ which contains more than one point and such that $\mathbb{C} \setminus K$ is connected. Let K' be the image of K under the map $z \to 1/z$. Then $\mathbb{C} \setminus K'$ is a simply connected set. So there is a conformal map f_K of $\mathbb{D}$ onto $\mathbb{C} \setminus K'$ with $f_K(0) = 0$ and $f_K'(0) > 0$. Define

$$F_K(z) = \frac{1}{f_K(1/z)} \tag{73}$$

Then F is a conformal map of $\mathbb{C} \setminus \overline{\mathbb{D}}$ onto $\mathbb{C} \setminus K$ with $\lim_{z\to\infty} F_K(z)/z = f_K'(0) > 0$. The *(logarithmic) capacity* of K is

$$cap(K) = -\log f_K'(0) = \log \lim_{z\to\infty} \frac{F_K(z)}{z} \tag{74}$$

Exercise: Show that if K_1 and K_2 are compact hulls with $K_1 \subset K_2$, then $cap(K_1) \leq cap(K_2)$. Show that if the two capacities are equal then $K_1 = K_2$. Hint: see the remarks on the proof of the Riemann mapping theorem.

The power series expanion of f_K about the origin shows that the Laurent expansion of F_K is of the form

$$F_K(z) = bz + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{75}$$

where $b = e^{cap(K)}$.

Proposition 7 *(Area theorem) Let K be a compact hull containing the origin with $cap(K) = 0$. Then the coeffecients in the Laurent expansion above satisfy*

$$area(K) = \pi \left[1 - \sum_{n=1}^{\infty} n|b_n|^2 \right] \tag{76}$$

In particular, $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

Proof: If γ is a simple, closed smooth curve, oriented clockwise and A is the region enclosed by γ, then by Green's theorem we have

$$\int_\gamma \overline{z}\, dz = \int_\gamma [(x\, dx + y\, dy) + i(x\, dy - y\, dx)] = \int_A 2i\, dx\, dy = 2i\, area(A) \tag{77}$$

Let $r > 1$ and let γ be the image of the circle of radius r under the map F_K. Then γ encloses a region which contains K and in the limit $r \to 1^+$, the area of the region enclosed by γ converges to the area of K. The curve γ is parameterized by $F_K(re^{i\theta})$ with $0 \leq \theta \leq 2\pi$. So

$$\int_\gamma \overline{z} dz = \int_0^{2\pi} \overline{F}_K(re^{i\theta})\, ire^{i\theta}\, F_K'(re^{i\theta})\, d\theta = 2\pi i [r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}] \tag{78}$$

Letting $r \to 1^+$ gives the proposition. ∎

Proposition 8 *If $f \in \mathcal{S}$, then there is an odd function $h \in \mathcal{S}$ such that $h(z)^2 = f(z^2)$ for $z \in \mathbb{D}$.*

Proof: Consider $f(z)/z$. Since $f(0) = 0$, the singularity at 0 is removable and so $f(z)/z$ is analytic. Since $f(z)$ is not zero for $z \neq 0$ and $f'(0) \neq 0$, $f(z)/z$ is not zero. Thus $f(z)/z$ has a square root, i.e., there is an analytic function $g(z)$ on $\mathbb{D}$ such that $g(z)^2 = f(z)/z$. Define $h(z) = zg(z^2)$. Then $h(z)$ is odd. And $h(z)^2 = z^2 g(z^2)^2 = f(z^2)$. Clearly, $h(0) = 0$ and $h'(0) = 1$. It remains to be shown that h is one to one. If $h(z_1) = h(z_2)$ then

$f(z_1^2) = f(z_2^2)$. Since f is one to one, this implies $z_1^2 = z_2^2$. So either $z_1 = z_2$ or $z_1 = -z_2$. The oddness of h rules out the second case. ■

Let $f \in \mathcal{S}$. Then the power series expansion of f about the origin is of the form

$$f(z) = z + \sum_{n=2} a_n z^n \tag{79}$$

The study of bounds on the a_n is a famous problem.

Proposition 9 *(Bierbach) If $f \in \mathcal{S}$ then $|a_2| \leq 2$.*

Proof: Let h be the odd function in $\mathcal{S}$ given by the above proposition so that $h(z)^2 = f(z^2)$ for $z \in \mathbb{D}$. Let $g(z) = 1/h(1/z)$. Then the Laurent expanion of $g(z)$ begins with

$$g(z) = z - \frac{a_2}{2z} + \cdots \tag{80}$$

So by the area theorem $|a_2| \leq 2$. ■

Bierbach conjectured that $|a_n| \leq n$ for all $n \geq 2$. This was proved by de Branges in 1985. (Note that $a_1 = 1$ by the definition of $\mathcal{S}$.)

Let $f \in \mathcal{S}$. Let $r < 1$. Then $f(r\mathbb{D})$ is an open set containing the origin. So it contains a disc about the origin. Amazingly, there is a lower bound on the radius of this disc which is uniform over $f \in \mathcal{S}$. We will use $B(a, r)$ to denote the disc of radius r centered at a.

Theorem 18 *(Koebe 1/4 theorem) If $f \in \mathcal{S}$ and $0 < r \leq 1$, then $B(0, r/4) \subset f(r\mathbb{D})$.*

Proof: We prove it for $r = 1$. The other values of r follow by considering the function $f(rz)/r$. Let $z_0 \notin f(\mathbb{D})$. We must show $|z_0| \geq 1/4$. Define

$$\hat{f}(z) = \frac{z_0 f(z)}{z_0 - f(z)} \tag{81}$$

This is a composition of $f(z)$ and a linear fractional transformation, so it is one to one. Let $f(z) = z + a_2 z^2 \cdots$. By the Bierbach theorem, $|a_2| \leq 2$. The expansion of $\hat{f}(z)$ is

$$\hat{f}(z) = z + (a_2 + \frac{1}{z_0}) z^2 + \cdots \tag{82}$$

Applying the Bierbach theorem to $\hat{f}$ we have $|a_2 + 1/z_0| \leq 2$. Combining this with $|a_2| \leq 2$ leads to $|z_0| \geq 1/4$. ■

Theorem 19 *(Growth and distortion theorems) Let $f \in \mathcal{S}$ and $z \in \mathbb{D}$. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \tag{83}$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \tag{84}$$

4.3 Half plane capacity

We continue to use $\mathbb{H}$ to denote the upper half plane. We do not include the real axis, so this is an open set.

Definition 16 *A bounded subset A of $\mathbb{H}$ is a "compact $\mathbb{H}$-hull" if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected.*

Proposition 10 *If A is a compact $\mathbb{H}$-hull, then there is a unique conformal map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ such that*

$$\lim_{z \to \infty} [g_A(z) - z] = 0 \tag{85}$$

Proof: The Riemann mapping theorem says there exist a conformal map g of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ which maps ∞ to itself. Since A is bounded it is contained in a ball $B(0, r)$ of radius r about 0 for some r. Consider $U = \{z : |z| > r\}$. On $\mathbb{H} \cap U$, g is an analytic function that can be continued to the boundary. It must map the boundary to the boundary of $\mathbb{H}$, i.e, the real axis. So by the Schwarz reflection principle, $g(z) = \overline{g(\overline{z})}$ defines an analytic continuation of g to all of U. So if we let $f(z) = 1/g(1/z)$, then f is analytic on $\{z : |z| < r\}$. $g(\infty) = \infty$ implies $f(0) = 0$. So the power series expansion of $f(z)$ about the origin is of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{86}$$

which implies g has an expansion about ∞ of the form

$$g(z) = b_{-1} z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n} \tag{87}$$

A little thought shows that since f maps parts of the real axis onto the real axis, all the b_i must be real. If we let $g_A(z) = (g(z) - b_0)/b_{-1}$, then g_A satisfies (85). We leave it to the reader to prove uniqueness. ∎

Definition 17 *Let A be a compact $\mathbb{H}$-hull. Let g_A be the unique conformal map given by the proposition. So the Laurent expansion is of the form*

$$g_A(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{88}$$

The half plane capacity, $hcap(A)$, is b_1.

Proposition 11 *Let $A_1 \subset A_2$ be compact $\mathbb{H}$-hulls. Then $hcap(A_1) \leq hcap(A_2)$ with equality only if the sets are equal. If A is a compact $\mathbb{H}$-hull and $r > 0, x \in \mathbb{R}$ then*

$$hcap(rA) = r^2\, hcap(A), \quad hcap(A + x) = hcap(A) \tag{89}$$

Exercise: Prove the above theorem.

Example: Let $g(z) = \sqrt{z^2+1}$. Then g is a conformal map of $\mathbb{H} \setminus (0, i]$ onto $\mathbb{H}$ which satisfies the normalization in the definition of half plane capacity. From $g(z) = z + 1/(2z) + \cdots$ we see that $hcap((0,i]) = 1/2$.

Exercise: The goal of this exercise is to see what happens to the capacity of the previous example when we rotate the segment $(0, i]$. Let $0 < \alpha < 1$ and define

$$f(z) = [z + 1 - \alpha]^\alpha [z - \alpha]^{1-\alpha} \tag{90}$$

Show that $f(z)$ conformally maps $\mathbb{H}$ onto $\mathbb{H} \setminus A$ where A is a line segment from 0 to $re^{i\alpha\pi}$ for some r. Let $g(z) = f^{-1}(z)$, so g conformally maps $\mathbb{H} \setminus A$ onto $\mathbb{H}$. Show that $g(z)$ satisfies (85). Find r as a function of α and from this compute the capacity of the segment of length 1 from 0 to $e^{i\alpha\pi}$. You can find the answers in example 3.37 in Lawler's book.

The next proposition gives an expression for the half plane capacity involving complex Brownian motion. We use Im to denote imaginary part.

Proposition 12 *Let A be a compact $\mathbb{H}$-hull. Let B_t be a complex Brownian motion starting at z. (z will be in the upper half plane). Let τ be the hitting time for $\mathbb{R} \cup A$. Let g_A be the conformal map used to define the half plane capacity of A. Then for $z \in \mathbb{H} \setminus A$,*

$$Im(z) = Im(g_A(z)) + E^z[Im(B_\tau)] \tag{91}$$

The half plane capacity is given by

$$hcap(A) = \lim_{y \to \infty} y E^{iy}[Im(B_\tau)] \tag{92}$$

If $rad(A) < 1$, then

$$hcap(A) = \frac{1}{\pi} \int_0^\pi E^{e^{i\theta}}[Im(B_\tau)] \sin(\theta)\, d\theta \tag{93}$$

Proof: In chapter 6 we will show that if ϕ is a bounded harmonic function, then $\phi(B_t)$ is a martingale. The real and imaginary parts of an analytic function are harmonic. Thus $\phi(z) = Im[z - g_A(z)]$ is a harmonic function, and so $\phi(B_t)$ is a martingale. It is also bounded. So by the optional sampling theorem,

$$E^z \phi(B_\tau) = E^z \phi(B_0) = \phi(z) = Im(z) - Im(g_A(z)) \tag{94}$$

Consider $\phi(B_\tau) = Im(B_\tau) - Im(g_A(B_\tau))$. By the definition of τ, B_τ is on the boundary of $\mathbb{H} \setminus A$. So $g_A(B_\tau)$ is on the bounday of $\mathbb{H}$, i.e., it is real and so $Im(g_A(B_\tau)) = 0$. This proves the first part of the theorem. Taking $z = iy$ in the above we have

$$E^{iy} Im B_\tau = y - Im(g_A(iy)) \tag{95}$$

As $y \to \infty$, $g_A(iy) = iy + hcap(A)/iy + O(1/y^2)$. So

$$y[y - Im(g_A(iy))] = hcap(A) + O(1/y) \tag{96}$$

The second statement in the proposition follows.

Now suppose $rad(A) < 1$. For $y > 1$, we start a Brownian motion at iy and let σ be the hitting time for the boundary of $\mathbb{H} \setminus \mathbb{D}$. Let $p(iy, z)$ be the probability density for B_σ where $z \in \partial(\mathbb{H} \setminus \mathbb{D})$. Then the strong Markov property implies

$$E^{iy}[Im(B_\tau)] = \int_0^\pi E^{e^{i\theta}}[Im(B_\tau)]\, p(iy, e^{i\theta})\, d\theta \tag{97}$$

It can be shown (lot of work) that on the boundary of the circle,

$$p(iy, e^{i\theta}) = \frac{2}{y\pi} \sin(\theta)[1 + O(y^{-1})], \quad 0 < \theta < \pi \tag{98}$$

which leads to the last equation in the proposition. ∎

The definition of half-plane capacity says that

$$g_A(z) = z + \frac{hcap(A)}{z} + O(1/z^2) \tag{99}$$

The next proposition will give some uniformity to this statement over A.

Proposition 13 *There is a constant c such that for all compact $\mathbb{H}$ hulls A, for $|z| \geq 2rad(A)$ we have*

$$|g_A(z) - z - \frac{hcap(A)}{z}| \leq c\frac{rad(A)hcap(A)}{|z|^2} \tag{100}$$

Proof: We prove it for $rad(A) = 1$ and leave the general case as an exercise. ∎

Exercise: Use a scaling argument and the theorem for $rad(A) = 1$ to prove the theorem for general A.

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4.4 Loewner differential equation

We recall one of the key ideas of SLE. Given a simple curve in the upper half plane starting at the origin, the half plane with the image of the curve up to time t removed is a simply connected domain. So there is a conformal map g_t from this "slit" half plane onto the half plane. If the curve is random, then this one parameter family of conformal maps g_t is random. SLE studies the random curve by studying the random conformal maps g_t. The following theorem says that g_t satisfies a differential equation known as Loewner's equation. Note that the theorem is deterministic; there is no randomness yet.

Theorem 20 *Let γ be a simple curve with $\gamma(0) = 0$ and $\gamma(0,\infty) \subset \mathbb{H}$. Let $b(t) = hcap(\gamma[0,t])$, and suppose that $b(t)$ is C^1 and $b(t) \to \infty$ as $t \to \infty$. Let $H_t = \mathbb{H} \setminus \gamma[0,t]$. This is a compact $\mathbb{H}$-hull, so we can let g_t be the unique conformal map of H_t onto $\mathbb{H}$ such that as $z \to \infty$,*

$$g_t(z) = z + \frac{hcap(\gamma[0,t])}{z} + O(\frac{1}{|z|^2}) \tag{101}$$

Then for $z \in \mathbb{H}$, $g_t(z)$ is the solution of

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z \tag{102}$$

where $U_t = g_t(\gamma(t))$. If $z \in \mathbb{H} \setminus \gamma(0,\infty)$, then the differential equation holds for all $t \geq 0$. It $z = \gamma(t_0)$, then the differential equation holds for $t < t_0$ and

$$\lim_{t \to t_0^-} g_t(z) = U_{t_0} \tag{103}$$

Partial Proof: The following proof has a couple of gaps. We will only consider a right sided derivative in (102) and we will need to make a continuity type assumption that while plausible is non-trivial to prove. The gaps may be filled in by consulting Lawler's book.

It can be shown that $\mathbb{C} \setminus \gamma[0,t]$ is locally connected. (This is a non-trivial baby analysis exercise.) So if we let $f_t = g_t^{-1}$, then f_t has a continuous extension to $\overline{\mathbb{H}}$. Let $U_t \in \mathbb{R}$ be such that $f_t(U_t) = \gamma(t)$. (How do you know U_t is unique?)

Fix $s > 0$. We want to compute

$$\dot{g_t}(z) = \lim_{\epsilon \to 0} \frac{g_{s+\epsilon}(z) - g_s(z)}{\epsilon} \tag{104}$$

We will only compute the right-sided derivative, i.e., we restrict to positive ϵ in the above limit. We look at the small portion of the curve from time s to $s + \epsilon$ and consider its image under g_s:

$$A = g_s(\gamma[s, s+\epsilon]) \tag{105}$$

This is a curve starting at the point U_s on the real axis. We have $g_{s+\epsilon} = g_A \circ g_s$. Letting $w = g_s(z)$, we have $g_{s+\epsilon}(z) - g_s(z) = g_A(w) - w$. We are going to apply (13) to g_A. The set A is very small, but it is not close to the origin. So we define $A' = A - U_s$. It is trivial to check that $g_{A'}(z) = g_A(z + U_s) - U_s$. We also note that $hcap(A') = hcap(A)$. The bound (13) says that for $|w| \geq 2rad(A')$ we have

$$|g_{A'}(w) - w - \frac{hcap(A)}{w}| \leq c\frac{rad(A')hcap(A)}{|w|^2} \tag{106}$$

and so

$$|g_A(w + U_s) - w - U_s - \frac{hcap(A)}{w}| \leq c\frac{rad(A')hcap(A)}{|w|^2} \tag{107}$$

By a trivial change of variables this gives

$$|g_A(w) - w - \frac{hcap(A)}{w - U_s}| \leq c\frac{rad(A')hcap(A)}{|w - U_s|^2} \tag{108}$$

for $|w - U_s| \geq 2rad(A')$. And so

$$|g_{s+\epsilon}(z) - g_s(z) - \frac{hcap(A)}{g_s(z) - U_s}| \leq c\frac{rad(A')hcap(A)}{|g_s(z) - U_s|^2} \tag{109}$$

for $|g_s(z) - U_s| \geq 2rad(A')$. We have

$$hcap(\gamma[0, s + \epsilon]) = hcap(\gamma[0, s]) + hcap(A) \tag{110}$$

So

$$\lim_{\epsilon \to 0^+} \frac{hcap(A)}{\epsilon} = \dot{b}(s) \tag{111}$$

We assume that $rad(A') \to 0$ as $\epsilon \to 0^+$. For $z \in \mathbb{H} \setminus \gamma[0, s]$, $g_s(z)$ will be in $\mathbb{H}$ and so we will have $|g_s(z) - U_s| \geq 2rad(A')$ for small enough ϵ. Thus we have proved that the Loewner equation (102) holds for the right sided derviative. ∎

What if $b(t) = hcap(\gamma[0, t])$ is not C^1 ? $b(t)$ is strictly increasing and it can be shown to be continuous (not trivial). So if we assume it goes to ∞ as $t \to \infty$, then it has a continuous inverse defined on $[0, \infty)$. Define

$$\tilde{\gamma(t)} = \gamma(b^{-1}(2t)) \tag{112}$$

Then $hcap(\tilde{\gamma(t)}) = 2t$. With this parameterization the Loewner equation is just

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z \tag{113}$$

We will call g_t arising from this equation a *Loewner chain with driving function* U_t. A g_t arising from the differential equation in the theorem will be called a *generalized Loewner chain.*

The real valued function U_t in the above theorem is known as the "driving function." The previous theorem starts with a simple curve and produces a driving function U_t. Now we go the other way. We assume we have some continuous real valued function U_t and see what we can get out of the Loewner equation. If U_t is only continuous, then we need not get a simple curve. The following theorem is a special case of theorem 4.5 in Lawler's book.

Theorem 21 *Let U_t be a continuous real-valued function on $[0, \infty)$. Let $b(t)$ be a C^1 real-valued function on $[0, \infty)$ which is increasing. For $z \in \mathbb{H}$, let $g_t(z)$ be the solution of*

$$\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z \tag{114}$$

The denominator can go to zero, causing the solution to fail to exist after some finite time. Define

$$T_z = \sup\{t : g_t(z) \, exists\} \tag{115}$$

Define

$$H_t = \{z \in \mathbb{H} : T_z > t\}, \quad K_t = \{z \in \mathbb{H} : T_z \leq t\} = \mathbb{H} \setminus H_t \tag{116}$$

Then *$g_t(z)$ is the unique conformal map of H_t onto $\mathbb{H}$ such that $g_t(z) - z \to 0$ as $z \to \infty$. Furthermore ,*

$$g_t(z) = z + \frac{b(t)}{z} + O(\frac{1}{|z|^2}), \quad z \to \infty \tag{117}$$

Proof: By standard ODE theorems, $g_t(z)$ is analytic in z and continuous in t when the solution exists. We show next that for each t, $g_t(z)$ is one to one.

Fix $z \neq w$ and let $\Delta_t = g_t(z) - g_t(w)$. From the dif. eq. we have

$$\dot{\Delta}_t = \frac{-\dot{b}(t)\Delta_t}{(g_t(z) - U_t)(g_s(z) - U_s)} \tag{118}$$

So

$$\frac{d \ln(\Delta_t)}{dt} = \frac{-\dot{b}(t)}{(g_t(z) - U_t)(g_t(w) - U_t)} \tag{119}$$

We solve this dif. eq. using the initial condition $\Delta_0 = z - w$. This gives

$$\Delta_t = (z - w) \exp\left[- \int_0^t \frac{-\dot{b}(s)}{(g_s(z) - U_s)(g_s(w) - U_s)} \, ds \right] \tag{120}$$

This shows $\Delta_t \neq 0$, and so g_t is one to one.

Next we need to show the range of g_t is $\mathbb{H}$. We split g_t into its real and imaginary parts, $g_t(z) = r_t(z) + iv_t(z)$ where r_t and v_t are real valued. Then we have

$$\dot{r_t}(z) = \frac{\dot{b}(t)(r_t(z) - U_t)}{(r_t(z) - U_t)^2 + v_t(z)^2} \tag{121}$$

and

$$\dot{v_t}(z) = \frac{-\dot{b}(t)v_t(z)}{(r_t(z) - U_t)^2 + v_t(z)^2} \tag{122}$$

Note that $v_t(z)$ is always decreasing and on the real axis $\dot{v_t(z)} = 0$. We leave it to the reader to show that $g_t(z)$ cannot leave the upper half plane. So $g_t(H_t)$ is a subset of $\mathbb{H}$.

To show $g_t(H_t)$ is all of $\mathbb{H}$ we consider the dif. eq. run backwards in time. Fix $t > 0$ and define $h_t(z)$ by

$$\dot{h_s}(w) = \frac{-\dot{b}(t-s)}{h_s(w) - U_{t-s}}, \quad h_0(w) = w \tag{123}$$

Now the imaginary part of $h_s(z)$ will always increase, so the solution to this differential equation will exist for all time, in particular for $0 \leq s \leq t$. Now $G_s = h_{t-s}$ satisfies the original Loewner equation and has $G_t(w) = h_0(w) = w$ and $G_0(z) = h_t(z)$. So $g_t(h_t(z)) = w$.

Finally, consider the dif. eq. for large z,

$$\dot{g_t}(z) = \frac{\dot{b}(t)}{g_t(z) - U_t} = \frac{\dot{b}(t)}{g_t(z)} + O(|g_t|^{-2}) = \frac{\dot{b}(t)}{z} + O(|z|^{-2}) \tag{124}$$

Since $g_0(z) = z$, this implies

$$g_t(z) - z = \int_0^t \dot{g_s}(z)\, ds = \int_0^t [\frac{\dot{b}(t)}{z} + O(|z|^{-2})]\, ds = \frac{b(t)}{z} + O(|z|^{-2}) \tag{125}$$

which verifies (117). ∎

Suppose K_t is a one parameter family of compact $\mathbb{H}$-hulls that is increasing in the sense that $K_t \subset K_s$ for $t < s$. Can we find a real valued driving function such that the above proposition produces the given K_t. The following example shows the answer is no.

Exercise: Define

$$H_1 = \{z \in \mathbb{H} : |z| > 1\} \tag{126}$$

The conformal map of H_1 onto $\mathbb{H}$ is

$$g_1(z) = z + \frac{1}{z} \tag{127}$$

Now define

$$K_t = \{z \in \mathbb{H} : |z| \leq t\} \tag{128}$$

Find g_t and show that there is no real valued U_t such that the Loewner equation produces this K_t.

In the above example, K_t grows at all points along its boundary. A natural question is what condition(s) must the K_t satisfy to come from a Loewner chain. One might guess that K_t must always grow locally, that is, the diameter of $K_{t+\delta} \setminus K_t$ must converge to zero as $\delta \to 0$. This is not true as the following example shows.

Example: Arc pinching off half disc.

Lemma 1 *Let g_t be the Loewner chain with driving function U_t. Define*

$$R_t = \max\{\sqrt{t}, \sup_{0 \le s \le t} |U_s|\} \tag{129}$$

Then $K_t \subset B(0, 4R_t)$. Moreover, if $|z| \ge 4R_t$, then

$$|g_s(z) - z| \le R_t, \quad for \quad 0 \le s \le t \tag{130}$$

Proof: Fix t. If $|z| \ge 4R_t$, then the bound (130) implies

$$|g_s(z) - U_s| = |(g_s(z) - z) + z - U_s| \ge |z| - |g_s(z) - z| - |U_s| \ge 4R_t - R_t - R_t = 2R_t \tag{131}$$

This shows that $g_s(z)$ is defined for $s \le t$. In fact, by continuity it shows it is defined for s slightly greater than t. So $z \in H_t$, i.e., $z \notin K_t$. Thus (130) implies $K_t \subset B(0, 4R_t)$. So it suffices to prove (130).

Suppose there is a time $s < t$ with $|g_s(z) - z| \ge R_t$. Let σ be the first such time. So $|g_s(z) - z| < R_t$ for $s \le \sigma$. By the above this implies $|\dot{g}_s(z)| \le 1/R_t$. Since $g_0(z) = z$, this implies that

$$|g_\sigma(z) - z| \le \frac{\sigma}{R_t} < \frac{t}{R_t} \le R_t \tag{132}$$

where we have used $t \le R_t^2$. But this contradicts $|g_\sigma(z) - z| \ge R_t$. ∎

The lemma says that initially K_t grows at the origin. Now we use the lemma to study the growth at other times. Fix $s > 0$. We saw that $K_{s+t} \setminus K_s$ need need not be small for small t. Consider instead $g_s(K_{s+t} \setminus K_s)$. We will use the lemma to show its diameter goes to zero as $t \to 0$. Let $\tilde{U}_t = U_{s+t}$, $\tilde{g}_t = g_{s+t} \circ g_s^{-1}$ Then $\tilde{g}_t$ satisfies the Loewner equation with driving function U_t and the initial condition $\tilde{g}_0(z) = z$. So the lemma says $\tilde{K}_t \subset B(0, \tilde{R}_t)$, where $\tilde{R}_t$ is defined using $\tilde{U}_t$. In particular $diam(\tilde{K}_t) \to 0$ as $t \to 0$. We claim that $\tilde{K}_t$ is $g_s(K_{s+t} \setminus K_s)$. By definition $\tilde{K}_t$ is the set of $z \in \mathbb{H}$ for which the solution $\tilde{g}_t(z)$ does not exist at time t. This is equivalent to saying that the solution g_{t+s} does not exist if we use the initial condition $g^{-1}(z)$. So

$$\tilde{K}_t = \{z \in \mathbb{H} : g_s^{-1}(z) \in K_{t+s}\} \tag{133}$$

Note that $g^{-1}(z)$ is always in H_t, i.e., not in K_t. So the above set equals

$$\{z \in \mathbb{H} : g_s^{-1}(z) \in K_{t+s} \setminus K_t\} = g_s(K_{t+s} \setminus K_t) \tag{134}$$

In fact there is a rather succint characterization of those K_t that can arise from a Loewner chain. A sketch of the proof of the following can be found in Werner's St. Flour notes, p. 23.

Theorem 22 *An increasing family K_t of compact $\mathbb{H}$-hulls comes from a Loewner chain if and only if the following are true. $hcap(K_t) = 2t$ for $t \geq 0$. And for all $T > 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that for $t \leq T$ there exists a bounded connected set $S \subset H_t$ with $diam(S) \leq \epsilon$ such that S disconnects $K_{t+\delta} \setminus K_t$ from infinity in H_t.*

The above has all been the chordal case. We state but do not prove the analog of theorem 21 for the radial case.

Theorem 23 *Let U_t be a continuous function on $[0, \infty)$ which takes values on the unit circle. Let $b(t)$ be a C^1 real-valued function on $[0, \infty)$ which is increasing. For $z \in \mathbb{H}$, let $g_t(z)$ be the solution of*

$$\dot{g}_t(z) = g_t(z)\dot{b}(t)\frac{U_t + g_t(z)}{U_t - g_t(z)}, \quad g_0(z) = z \tag{135}$$

Define

$$T_z = \sup\{t : g_t(z)\, exists\} \tag{136}$$

Define

$$D_t = \{z \in \mathbb{D} : T_z > t\} \tag{137}$$

Then $g_t(z)$ is the unique conformal map of D_t onto $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. Furthermore ,

$$\ln(g_t'(0)) = b(t) \tag{138}$$

Theorems 21 and 23 produce increasing families of sets, K_t. If the driving function is nice enough, K_t will be a simple curve. In the chordal case U_t will be $\sqrt{\kappa}B_t$ where B_t is a standard Brownian motion. Interestingly, it turns out the smoothness of Brownian motion is right at the borderline of "nice enough." So the deterministic theorems about the Loewner equation will not say anything about SLE. As we will see, whether K_t is a simple curve depends on the value of κ. Even when K_t is not a simple curve, it will turn out that is generated by a non-simple curve in the following sense.

Definition 18 *We will say that the Loewner chain g_t is generated by the curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ if for each t, the domain H_t of g_t is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.*

For example, when $\kappa = 6$, K_t is not a simple curve. The percolation exploration process corresponds to the curve that generates K_t.

We end this chapter with a couple of examples of U_t for which the Loewner equation may be solved explicitly.

If $g_t(z) = \sqrt{z^2 + 4t}$, then g_t solves the Loewner equation with $U_t = 0$. The hull K_t is just the vertical line segment from 0 to $2ti$.

Let $0 < \alpha < 1$. Define

$$\phi(z) = [z + (1 - \alpha)]^\alpha [z - \alpha]^{1-\alpha} \tag{139}$$

Then ϕ is a conformal map of $\mathbb{H}$ onto $\mathbb{H}$ minus a line segment starting at 0 and forming an angle of $(1 - \alpha)\pi$ with the positive real axis. The length of the segment depends on α. By computing this length one finds that if we let

$$f_t(z) = \sqrt{\frac{4t}{\alpha(1-\alpha)}}\phi(z\sqrt{\frac{\alpha(1-\alpha)}{4t}}) \tag{140}$$

and let $g_t = f_t^{-1}$ then g_t satisfies the Loewner equation and K_t is a line segment starting at 0 and forming an angle of $(1-\alpha)\pi$ with the positive horizontal axis whose length grows with t. The driving function works out to be $U_t = c_\alpha t$ where $c_\alpha = 2(2\alpha - 1)\sqrt{\alpha(1 - \alpha)}$. Note that c_α is positive for $\alpha < 1/2$ while it is negative for $\alpha > 1/2$.