6 Stochastic differential equations

6.1 Notation and some definitions

This chapter will follow the first chapter in Lawler's book very closely. Accordingly we will state theorems but will not reproduce the proofs which may be found in his book. We will add some commentary and background.

We want to define integration with respect to the Brownian path. We start with some remarks as to why this must be a new kind of integral and at the same time introduce some notation.

A partition Π of a time interval $[0,s]$ is a finite set of times $t_0 < t_1 < t_2 < \cdots < t_n$ with $t_0 = 0$ and $t_n = s$. The "gap" of the partition $||\Pi||$ is the length of the largest subinterval in the partition. Given such a partition we pick times $\overline{t_i}$ in each interval $[t_{i-1}, t_i]$ and then define the Riemann sum for a function f on $[0,t]$ by

$$\sum_{i=1}^{n} f(\overline{t_i})(t_i - t_{i-1}) \tag{150}$$

If f is Riemann integrable, then these Riemann sums will converge to $\int_0^s f(x)dx$.

Now suppose $F(t)$ is an increasing function. Then we define the Riemann-Stieljes sum by

$$\sum_{i=1}^{n} f(\overline{t_i})(F(t_i) - F(t_{i-1})) \tag{151}$$

Of course, if $F(t) = t$ this reduces to the Riemann sum. The Riemann-Stieljes integral is defined as the limit of this sum. The function $F(t)$ need not be increasing for this integral to make sense, but it cannot be arbitrary. It must have bounded variation. This means that

$$\sup_{\Pi} \sum_{i=1}^{n} |F(t_i) - F(t_{i-1})| < \infty \tag{152}$$

We want to define $\int_0^s f(t)dB_t$ where B_t is a Brownian motion. However, B_t does not have bounded variation, so we cannot use the Riemann-Stieljes integral.

We recall some definition and introduce some new ones. A filtration $\mathcal{F}_t$ is an increasing sequence of σ-fields. A stochastic process H_t is *adapted* (to $\mathcal{F}_t$) if for all t, H_t is measureable with respect to $\mathcal{F}_t$. We assume that we have a filtration $\mathcal{F}_t$ such that our Brownian motion is adapted to it. We will say that the process is *continuous* if H_t is a continuous function of t with probability one. We will say it is *bounded* if there is a constant $M < \infty$ such that $|H_t| \leq M$ for all t with probabilty one. We say H_t is a *martingale* if for all t, $E|H_t| < \infty$ and $E[H_t|\mathcal{F}_s] = H_s$ for $s \leq t$. We say H_t is a *square integrable martingale* if H_t is a martingale with $EH_t^2 < \infty$ for all t. Finally we say H_t is a *local martingale* if there is a sequence of stopping times $\tau_1 < \tau_2 < \cdots$ with respect to $\mathcal{F}_t$ such that $\tau_j \to \infty$ as $j \to \infty$ with probability one and for each j, $H_{\min\{t,\tau_j\}}$ is a martingale.

6.2 Definition of integration with respect to Brownian motion

This section follows Lawler's section 1.2 closely. We will state the main results and refer the reader to Lawler for proofs.

The stochastic process H_t is *simple* if it is of the form

$$H_s = \sum_{j=1}^{n} X_j 1_{[t_{j-1},t_j)}(s) \tag{153}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and for each j, X_j is a bounded random variable that is measureable with respect to $\mathcal{F}_{t_{j-1}}$. NB: it is not just measurable with respect to $\mathcal{F}_{t_j}$. So X_j is independent of the increments of B_t within the time interval $[t_{j-1}, t_j]$. And so $\int_{t_{j-1}}^{t_j} X_j \, dB_s$ should just be $X_j(B_{t_j} - B_{t_{j-1}})$. This motivates defining

$$\int_0^t H_s \, dB_s = \sum_{j=1}^{k-1} X_j(B_{t_j} - B_{t_{j-1}}) + X_k(B_t - B_{k-1}) \tag{154}$$

where k is such that $t \in [t_{k-1}, t_k]$. We will denote $\int_0^t H_s \, dB_s$ by Z_t. It is a stochastic process.

Definition 22 *The quadratic variation of a process*

$$Z_t = \int_0^t H_s \, dB_s \tag{155}$$

is defined by

$$< Z >_t = \int_0^t H_s^2 \, ds \tag{156}$$

Note that $< Z >_t$ is a stochastic process. The notation is standard, but a bit strange. Physicists often use $< \ >$ to denote expectation and that is not what it means in the above context. Also, the quadratic variation is defined using H_s not Z_t. Finally we emphasize that the integral in the definition of $< Z >_t$ is just an ordinary Riemann integral; it is not integration with respect to dB_s.

Proposition 15 *Let H_s and K_s be simple adapted processes, and $a, b \in \mathbb{R}$. Then $aH_s + bK_s$ is a simple adapted process and*

$$\int_0^t (aH_s + bK_s) \, dB_s = a \int_0^t H_s \, dB_s + b \int_0^t K_s \, dB_s \tag{157}$$

We define

$$Z_t = \int_0^t H_s \, dB_s \tag{158}$$

Then Z_t is a square integrable martingale. Also, $Z_t^2 - < Z >_t$ is a martingale. In particular

$$EZ_t^2 = E < Z >_t = \int_0^t EH_s^2 \, ds \tag{159}$$

Exercise: Prove that Z_t is a martingale. There are a lot of cases to be checked. I recommend checking the following. Take H_s to just have a single "step",i.e., assume it is of the form

$$H_s = X 1_{[t_0,t_1)}(s) \tag{160}$$

where X is measureable with respect to $\mathcal{F}_{t_0}$. Check that for $t > s$

$$E[Z_t | \mathcal{F}_s] = Z_s \tag{161}$$

for two cases: $t_0 \leq s < t \leq t_1$ and $s \leq t_0 \leq t \leq t_1$. Hints: For the first case, X is measurable with respect to $\mathcal{F}_s$ (Why?). For the second case you will need to use the fact that conditioning on $\mathcal{F}_{t_0}$ and then on $\mathcal{F}_s$ is the same as just conditioning as $\mathcal{F}_s$.

Exercise: Prove that $Z_t^2 - < Z >_t$ is a martingale. Again, there are lots of cases to check. Just do it for H_s of the form (160).

Proposition 16 *Let H_s be a simple adapted process. Let*

$$Z_t = \int_0^t H_s \, dB_s \tag{162}$$

Let Π_n be a sequence of partitions of $[0, t]$: $0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k_n}^n = t$. Suppose that $||\Pi_n|| \to 0$. Then

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} (Z_{t_j} - Z_{t_{j-1}})^2 = < Z >_t \tag{163}$$

where the convergence is in L^2. If $\sum_{n=1}^\infty ||\Pi_n|| < \infty$, then the convergence is also with probabilty one.

Proof: See Lawler.

We now extended the definition of $\int_0^t H_s \, dB_s$ to non-simple processes H_s. Various notions of convergence of a sequence of random variables will be running around, so we briefly review them.

Definition 23 *Let X_n be a sequence of random variables, X a random variable. We say that X_n converges to X with probabilty one if*

$$P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1 \tag{164}$$

We say X_n converges to X in probability if for every $\epsilon > 0$ we have

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0 \tag{165}$$

Finally we say that X_n converges to X in L^p ($p \geq 1$) if

$$\lim_{n\to\infty} E|X_n - X|^p = 0 \tag{166}$$

It is not hard to show that convegence with probability one implies convergence in probability and that for any $p \geq 1$ convegence in L^p implies convegence in probability. All other possible implications are not true. However, if X_n converges to X in L^p then one can find a subsequence that converges with probability one. We now return to the stochastic integral.

Proposition 17 *Let H_s be an adapted process which is bounded and continuous in s. Fix a $t > 0$. Let*

$$\Pi_n = \{0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k_n}^n = t\} \tag{167}$$

with $||\Pi_n|| \to 0$. Then

$$\sum_{j=1}^{k_n} H_{t_{j-1}}(B_{t_j} - B_{t_{j-1}}) \tag{168}$$

is a Cauchy sequence in L^2. We define $Z_t = \int_0^t H_s\, dB_s$ to be its limit in L^2. We define the quadratic variation as before

$$< Z >_t = \int_0^t H_s^2\, ds \tag{169}$$

Then Z_t and $Z_t^2 - Z_t$ are martingales.

The proof may be found in Lawler. At this point we could use the proposition to define Z_t for all t. The problem with this approach is that we would like to construct Z_t so that it is continuous as a function of t and using the above proposition to define it may screw up this property. Thus we will be forced into a bit of a "detour." First, we work out an example.

Example : We are going to compute $\int_0^t B_s\, dB_s$. In this example $H_s = B_s$. Given a sequence of partitions, H_s^n is defined to be $B_{t_{j-1}}$ where $s \in [t_{j-1}, t_j)$ So

$$\int_0^t H_s^n\, dB_s = \sum_{j=1}^{k_n} B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}}) \tag{170}$$

We denote this quantity by L. The stochastic integral we are trying to compute is its limit in L^2. Now we define

$$R = \sum_{j=1}^{k_n} B_{t_j}(B_{t_j} - B_{t_{j-1}}) \tag{171}$$

We can think of L and R as left and right Riemann sums. We have

$$R + L = \sum_{j=1}^{k_n}(B_{t_j} + B_{t_{j-1}})(B_{t_j} - B_{t_{j-1}}) = \sum_{j=1}^{k_n}(B_{t_j}^2 - B_{t_{j-1}}^2) = B_t^2 - B_0^2 \tag{172}$$

and we have

$$R - L = \sum_{j=1}^{k_n}(B_{t_j} - B_{t_{j-1}})^2 \tag{173}$$

This converges to t. Thus

$$2L = (R + L) - (R - L) \to B_t^2 - B_0^2 - t \tag{174}$$

So we have shown that

$$\int_0^t B_s\, dB_s = \frac{1}{2}(B_t^2 - B_0^2 - t) \tag{175}$$

Note that this is a martingale as it should be.

We defined Z_t using an L^2 limit above. We want to make this a point-wise limit. For each t we can find a subsequence of the sequence of partitions such that the limit is with probability one. However, each t may require a different subsequence. We can do it for a countable set of t's by a diagonalization argument. We do this for the dyadic t (rational t with a denominator that is a power of 2).

Proposition 18 *Let Π_n be a sequence of partitions with $||\Pi_n|| \to 0$ such that for all dyadic t,*

$$Z_t = \lim_{n\to\infty} \sum_{j=1}^{k_n} H_{t_{j-1}}(B_{t_j} - B_{t_{j-1}}) \tag{176}$$

where the convergence is both in L^2 and with probability one. Then with probability one, for each $t > 0$, $s \to Z_s$ is a uniformly continuous function on $D \cap [0, t]$. Here D denotes the set of dyadic rationals.

We refer the reader to Lawler for the proof. Now $D \cap [0, t]$ is dense in $[0, t]$ and so a uniformly continuous function on $D \cap [0, t]$ has a unique continuous extension to $[0, t]$. We define Z_s for non-dyadic s by this extension. It is not hard to show that for all t and any sequence of partitions Π_n of $[0, t]$ with $||\Pi_n|| \to 0$ we have the convergence in (176) in L^2.

The final step is to remove the restriction that H_s must be bounded. The construction is similar to the way you extend the definition of the Lebesgue integral from bounded functions to more general functions. Let H_s^N be H_s when $|H_s| \leq N$, $H_s^N = N$ when $H_s > N$, and $H_s^N = -N$ when $H_s < N$. You then construct $\int_0^t H_s dB_s$ as a limit of $\int_0^t H_s^N dB_s$ for $N \to \infty$. We refer to Lawler for the details.

When H_s is unbounded, the resulting Z_t may not be integrable and so need not be a martingale. It is however a local martingale. See Lawler for more on this.

Proposition 19 *If H_s and K_s are continuous, adapted processes and $a, b \in reals$, then*

$$\int_0^t (aH_s + bK_s)\, dB_s = a \int_0^t H_s\, dB_s + b \int_0^t K_s\, dB_s \tag{177}$$

If

$$\int_0^t E[H_s^2] ds < \infty \tag{178}$$

for all t, then Z_t is a square integrable martingale and $Z_t^2 - < Z >_t$ is a martingale.

Given a sequence of partitions Π_n you can show there is a subsequence such that with probability one, for all $s \leq t$ we have

$$\lim_{n\to\infty} \sum_{t_j \leq s} H_{t_{j-1}}\, (B_{t_j} - B_{t_{j-1}}) = \int_0^s H_r\, dB_r \tag{179}$$

This is not obvious. See Lawler for the argument. We have

Proposition 20 *Let H_s be bounded, adapted process, Π_n a sequence of partitions as above. Then*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} (Z_{t_j} - Z_{t_{j-1}})^2 = < Z >_t = \int_0^t H_s^2\, ds \tag{180}$$

If G_s is any continuous process then

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} G_{s_j}\, (Z_{t_j} - Z_{t_{j-1}})^2 = < Z >_t = \int_0^t G_s\, H_s^2\, ds \tag{181}$$

where s_j is any point in $[t_{j-1}, t_j]$.

Remark: As a function of t, $< Z >_t$ is differentiable with

$$\frac{d < Z >_t}{dt} = H_t^2 \tag{182}$$

last integral in the above proposition is often written as

$$\int_0^t G_s\, H_s^2\, ds = \int_0^t G_s\, d < Z >_s \tag{183}$$

where the intergal on the right is a Riemann-Stieltjes integral.

6.3 Ito's formula

Suppose that H_s is an adapted, continuous process and

$$Z_t = \int_0^t H_s \, dB_s \tag{184}$$

We can imagine writing dB_s as $\frac{dB_s}{ds} ds$ and then differentiating this equation with respect to t to get

$$\frac{dZ_t}{dt} = H_t \frac{dB_t}{dt} \tag{185}$$

This is usually written as

$$dZ_t = H_t \, dB_t \tag{186}$$

We emphasize that (186) is nothing more than a shorthand for (184).(186) is a simple example of a stochastic differential equation and its solution Z_t is given by (184).

Ito's formula is a chain rule for stochastic "differentiation."

Proposition 21 *(Ito's formula) Let H_s be a continuous, adapted process. Let $h(t,x)$ be a function on $\mathbb{R}^2$ that is C^1 in t and C^2 in x. Let*

$$Z_t = \int_0^t H_s \, dB_s \tag{187}$$

Then

$$h(t, Z_t) - h(0, Z_0) = \int_0^t h'(s, Z_s) \, H_s \, dB_s + \int_0^t [\dot{h}(s, Z_s) + \frac{1}{2} h''(s, Z_s) \, H_s^2] \, ds \tag{188}$$

where $\cdot$ denotes differentiation with respect to time t and $'$ denotes differentiation with respect to space x.

For the proof we refer to Lawler. He states and proves a more general version in which $h(t,x)$ is random as well. Here are some hueristics. We can rewrite the above as

$$dh(t, Z_t) = h'(t, Z_t) \, H_t \, dB_t + \dot{h}(t, Z_t) dt + \frac{1}{2} h''(t, Z_t) \, H_t^2 \, dt \tag{189}$$

We emphasize yet again that (189) is nothing but a shorthand for (188). What is strange about this formula is the last term which seems to be second order. It arises because the typical size of $B_{t+dt} - B_t$ is of order $\sqrt{dt}$. It is often said that all you need to remember to do stochastic calculus is that

$$(dB_t)^2 = dt \tag{190}$$

This can be used to give a highly non-rigorous derivation of Ito's formula as follows. We try to Taylor expand $h(t+dt, Z_{t+dt})$. Z_{t+dt} is $Z_t + H_t\,dB_t$. Since $(dB_t)^2$ is of order dt, to expand to first order in dt we must expand to second order in dB_t. So

$$h(t+dt, Z_{t+dt}) = h(t+dt, Z_t + H_t\,dB_t) \tag{191}$$

$$= h(t, Z_t) + \dot{h}(t, Z_t)dt + h'(t, Z_t)\,H_t\,dB_t + \tfrac{1}{2}h''(t, Z_t)\,H_t^2\,(dB_t)^2 \tag{192}$$

Replacing the last $(dB_t)^2$ by dt gives (189).

Example: Let $h(t,x) = x^2$, and $H_s = 1$. Then $Z_t = B_t$ and Ito says

$$B_t^2 - B_0^2 = \int_0^t 2B_s\,dB_s + \frac{1}{2}\int_0^t 2\,ds \tag{193}$$

which is the formula we derived before by brute force:

$$\int_0^t B_s\,dB_s = \frac{1}{2}(B_t^2 - B_0^2 - t) \tag{194}$$

We can write this as $d(B_t)^2 = 2dB_t + 2dt$ This shows that the solution to the stochastic differential equation $dY_t = dB_t + dt$ is $Y_t = B_t^2/2$.

Example: Consider the ordinary differential equation

$$\frac{dy}{dt}(t) = h(t)y(t) \tag{195}$$

where $h(t)$ is some "known" function. Its solution is $y(t) = exp(\int_0^t h(s)ds)$. Now consider the stochastic differential equation

$$dY_t = H_t\,Y_t\,dB_t \tag{196}$$

Letting $Z_t = \int_0^t H_s\,dB_s$, our first guess might be that $Y_t = \exp(Z_t)$ is the solution. This is not correct. Taking $h(t,x) = e^x$, Ito says

$$\exp(Z_t) - 1 = \int_0^t \exp(Z_s)\,H_s\,dB_s + \frac{1}{2}\int_0^t \exp(Z_s)\,H_s^2\,ds \tag{197}$$

We claim that the solution of the above equation is actually

$$Y_t = \exp(Z_t - < Z >_t /2) \tag{198}$$

Recall that $< Z >_t$ is defined by an ordinary integral, $\int_0^t H_s^2\,ds$. Let

$$h(t,x) = \exp(x - < Z >_t /2) \tag{199}$$

This is a random function. So we need the more general version of Ito found in Lawler's book. Note that $< Z >_t$ is C^1 in t, and

$$\dot{h}(t,x) = -\frac{1}{2} H_t^2 \, h(t,x) \tag{200}$$

So Ito says

$$\begin{aligned} \exp(Z_t - < Z >_t /2) - 1 = \tfrac{-1}{2} \textstyle\int_0^t \exp(Z_s - < Z_s/2) \, H_s^2 \, ds \\ + \textstyle\int_0^t \exp(Z_s - < Z >_s /2) \, H_s \, dB_s + \tfrac{1}{2} \textstyle\int_0^t \exp(Z_s - < Z_s > /2) \, ds \\ = \textstyle\int_0^t \exp(Z_s - < z >_t /2) \, H_s \, dB_s \end{aligned}$$

Thus

$$Y_t = \exp(Z_t - < Z >_t /2) \tag{201}$$

Note that (201) shows that Y_t is a stochastic integral and so is a martingale. In other words, the fact that the dt terms in the above calculation cancelled out tells us that Y_t is a martingale.

Exercise: Let $a \in reals$. Show that

$$Y_t = \exp(aZ_t - a^2 < Z >_t /2) \tag{202}$$

and

$$W_t = \exp(iaZ_t + a^2 < Z >_t /2) \tag{203}$$

are martingales. (This should be straightforward calculation.)

So far we have been considering integration with respect to a 1d Brownian motion. Now let $B_t = (B_t^1, B_t^2, \cdots, B_t^d)$ be a standard d-dimensional Brownian motion. Then for each i we can define $Z_t^i = \int_0^t H_s \, dB_s^i$. Note that for $i \neq j$, Z_t^i and Z_t^j have the same distribution, but they are not equal. Note also that even though B_t^i and B_t^j are independent, this does not imply Z_t^i and Z_t^j are independent because they both depend on the stochastic process H_s. Ito's formula generalize to multidimensional Brownian motion. We state the special case of $H_s = 1$.

Proposition 22 *(Ito's formula in d dimensions) Let H_s be a continuous, adapted process. Let $h(t, x_1, \cdots, x_n)$ be C^1 in t and C^2 in $(x_1, x_2, \cdots, x_n)$. Let B_t be a standard d dimensional Brownian motion. Then*

$$h(t, B_t) - h(0, B_0) = \sum_{j=1}^d \int_0^t h_j(s, B_s) \, dB_s^j + \int_0^t [\dot{h}(s, B_s) + \frac{1}{2} \Delta h(s, B_s)] \, ds \tag{204}$$

where $\cdot$ denotes differentiation with respect to time t, h_j is the partial derivative of h with respect to x_j and Δ is the d-dimensional Laplacian.

Example (martingales from harmonic functions): Let f be a C^2 function on a domain in $\mathbb{R}^d$. We say f is harmonic if it satisfies Laplaces equation, $\Delta f = 0$. Let B_t be a standard d-dimensional Brownian motion and f harmonic. Then Ito says

$$f(B_t) - f(B_0) = \sum_{j=1}^{d} \int_0^t f_j(s, B_s)\, dB_s^j \tag{205}$$

There is not ds integral in the above, so this shows that $f(B_t)$ is a martingale. Actually, it only shows that it is a local martingale. To insure that $f(B_t)$ is integrable we need some assumption on f. Suppose D is an open set in $\mathbb{R}^d$ and f is harmonic on D. Let τ_D be the exit time, i.e.,

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\} \tag{206}$$

Then $Y_{\min\{t,\tau_D\}}$ is a local martingale. If f is bounded on D, then it is a bounded martingale.

Example (SLE and the Bessel process): Recall the SLE equation:

$$\frac{dg_t}{dt} = \frac{2}{g_t - \sqrt{\kappa} B_t} \tag{207}$$

Let

$$Y_t = g_t - \sqrt{\kappa} B_t \tag{208}$$

Think of this as a complex valued stochastic process with z as a parameter. Formally,

$$\frac{dY_t}{dt} = \frac{2}{Y_t} - \sqrt{\kappa} \frac{dB_t}{dt} \tag{209}$$

or

$$dY_t = \frac{2}{Y_t} dt - \sqrt{\kappa} dB_t \tag{210}$$

If we let $X_t = Y_t/\sqrt{\kappa}$, this becomes

$$dX_t = \frac{2}{\kappa} \frac{dt}{X_t} - dB_t \tag{211}$$

This is a well known stochastic differential equation whose solution is known as the Bessel process. It is usually written in the form

$$dX_t = a \frac{dt}{X_t} + dB_t \tag{212}$$

This is equivalent to the above (with $a = 2/\kappa$) since B_t and $-B_t$ have the same distribution.

As always, the real meaning of (212) is the integral equation:

$$X_t - X_0 = a \int_0^t \frac{ds}{X_s} + B_t \tag{213}$$

It is trivial to check that this is equivalent to the integral form of the SLE differential equation.

6.4 Time change of martingales

Recall that

$$Z_t = \int_0^t H_s \, dB_s \tag{214}$$

is a martingale. Of course, the simplest case of this is the martingale B_t. The following theorem says that if we do a random time change of Z_t then we can turn it into Brownian motion.

Let $(B_t^1, \cdots, B_t^d)$ be a standard d-dimensional Brownian motion. Let H_t^j be adapted, continuous processes. Define

$$Z_t = \sum_{j=1}^{d} \int_0^t H_s^j \, dB_s^j \tag{215}$$

and

$$< Z >_t = \sum_{j=1}^{d} \int_0^t (H_s^j)^2 \, ds \tag{216}$$

Proposition 23 *Assume that*

$$\sum_{j=1}^{d} \int_0^\infty (H_s^j)^2 \, ds = \infty \tag{217}$$

Define stopping times by

$$\tau_r = \inf\{t :< Z >_t = r\} \tag{218}$$

and then define $W_r = B_{\tau_r}$. Then W_r is a standard Brownian motion with respect to the filtration $\mathcal{F}_{\tau_r}$.

Proof: We will prove that W_r is a normal random variable with mean zero and variance r. We refer the reader to Lawler for the rest of the proof. Let

$$Y_t = \exp[iyZ_t + y^2 < Z >_t /2] \tag{219}$$

Ito shows that this is local martingale. For $t \leq \tau_r$, Y_t is bounded by $\exp(y^2 r/2)$. By the optional sampling theorem, $E[Y_{\tau_r}] = E[Y_0] = 1$. So

$$E[iyZ_{\tau_r}] = \exp(-y^2 r/2) \tag{220}$$

This shows that W_r is normal with mean zero and variance r.

In the above proposition we assumed that the martingale Z_t came from a stochastic integral. There is a more general version of this theorem. We give a glimpse of it without defining all the terms we use. Let X_t be a square integral martingale. Then X_t^2 is a submartingale (undefined). The Doob-Meyer decomposition says we can write it as a sum $X_t^2 = M_t + A_t$ where M_t is a martingale and A_t is increasing as a function of t. We then define the quadratic variation of Z_t to be A_t. (Note that if Z_t comes from a stochastic integral this agrees with our original definition.)

Theorem 25 *Let M_t be a continuous square integrable martingale with respect to the filtration $\mathcal{F}_t$. Suppose that* $\lim_{t\to\infty} < M >_t = \infty$. *Define*

$$\tau_s = \inf\{t \geq 0 :< M >_t > s\} \tag{221}$$

Then $X_s = M_{\tau_s}$ is a standard Brownian motion.

A precise statement and proof can be found on p. 174 of Karatzas and Shreve (*Brownian motion and stochastic calculus*).

The above theorem can be used to fill in a gap at the end of our "derivation" of SLE. Recall that we showed that the driving function U_t is a continuous process with independent, stationary increments. It is easy to show that independent increments and the mean of the process being zero imply it is a martingale. (The reader should check this!). One then needs to argue that the time change that makes it into standard Brownian motion can only be of the form $\tau = ct$, for some deterministic constant c. So U_t is just a constant times standard Brownian motion.

We return to the Bessel process. Let $(B_t^1, \cdots, B_t^d)$ be a standard d-dimensional Brownian motion. Let

$$X_t = h(B_t^1, \cdots, B_t^d) \tag{222}$$

where $h(x_1, \cdots, x_d) = \sqrt{x_1^2 + \cdots + x_d^2}$. Some calculus shows $\Delta h = (d-1)/h$. So Ito says

$$dX_t = \sum_j B_j X_t^{-1} dB_t^j + \frac{d-1}{2X_t} dt \tag{223}$$

Let

$$\bar{B}_t = \sum_j \int_0^t \frac{B_s^j}{X_s} dB_s^j \tag{224}$$

A simple computation shows $< \bar{B} >_t = t$. By the change of time proposition we conclude that $\bar{B}_t$ is a standard Brownian motion. So

$$dX_t = d\bar{B}_t + \frac{d-1}{2X_t} dt \tag{225}$$

Thus X_t satisfies the Bessel SDE with $a = (d-1)/2$.

Conformal invariance of BM Let $B_t = B_t^1 + iB_t^2$ be a complex Brownian motion starting at some $z \in \mathbb{C}$. Let D be a domain in the complex plane containing z. Let f be a conformal map on D and $D' = f(D)$. Define the exit time

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\} \tag{226}$$

If we apply the conformal map f to the Brownian path B_t, $0 \leq t \leq \tau_D$, then we get a path in D' from $f(z)$ to the boundary of D'. The following proposition says that if we look at this path and ignore its parameterization, then it has the same distribution as a complex Brownian motion started at $f(z)$ and run until it exits D'.

Proposition 24 *Let*

$$S_t = \int_0^t |f'(B_s)|^2 \, ds, \quad 0 \le t \le \tau_D \tag{227}$$

Define σ_t *by* $S_{\sigma_t} = t$*, and then define* $Y_s = f(B_{\sigma_s})$*. Then* Y_s *is a standard Brownian motion starting at* $f(z)$ *until it exits* D'*.*

Proof: Write f as $u + iv$. Then u and v are harmonic functions, and so $u(B_t)$ and $v(B_t)$ are local martingales. Ito says

$$\begin{aligned} du(B_t) &= u_x(B_t) dB_t^1 + u_y(B_t) dB_t^2 \\ dv(B_t) &= v_x(B_t) dB_t^1 + v_y(B_t) dB_t^2 \end{aligned} \tag{228}$$

By the Cauchy Riemann equations the last equation can be rewritten as

$$dv(B_t) = -u_y(B_t) dB_t^1 + u_x(B_t) dB_t^2 \tag{229}$$

The quadratic variance of $u(B_t)$ is

$$< u(B) >_t = \int_0^t \left[u_x(B_s)^2 + u_y(B_s)^2 \right] \, ds \tag{230}$$

Now $f' = u_x + iv_x = u_x - iu_y$ using CR again. So

$$< u(B) >_t = \int_0^t |f'(B_s)|^2 \, ds \tag{231}$$

Thus the time change in the proposition is precisely the time change needed to make $u(B_{\sigma_t})$ into a standard Brownian motion. Call it $\hat{B}_t^1$. Likewise, $\hat{B}_t^2 = v(B_{\sigma_t})$ is a standard Brownian motion.

It remains to be shown that $\bar{B}_t^1$ and $\bar{B}_t^2$ are independent. Here are the highlights; details left to the reader. We known each of them is Gaussian. We claim they are jointly Gaussian, i.e., any linear combination is Gaussian. This follows from the martingale argument above. To show jointly Gaussian RV's are independent all you need to show is that their covariance is zero. So we need to show $E[\bar{B}_t^1 \bar{B}_s^2] = 0$. This follows from the first equation in (228) and (229). ∎

We end this section with a little more on Ito that we will need in the next section. A continuous *semimartingale* is a stochastic process of the form

$$Z_t = R_t + \int_0^t H_s \, dB_s \tag{232}$$

where H_s, R_s are continuous, adapted processes and R_t has bounded variation. Note that bounded variation is enough to define a Riemann-Stieljes integral with respect to dR_t. The shorthand for the above is

$$dZ_t = dR_t + H_s\, dB_s \tag{233}$$

The quadratic variation of Z_t is defined to be the quadratic variation of the stochatic integral in Z_t. (The quadratic variation of a process with bounded variation is zero.) We will state a version of Ito formula for a semimartingale. We first give a formal derivation:

Let $h(t,x)$ be the usual. Consider $h(t, Z_t)$.

$$dh(t, Z_t) = \frac{\partial h}{\partial t}\, dt + \frac{\partial h}{\partial x}\, dZ_t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}\,(dZ_t)^2 \tag{234}$$

We interpret $(dZ_t)^2$ to be $H_t^2 dt$, i.e., $d < Z >_t$.

Proposition 25 *Z_t is as above. Let $h(t,x)$ be C^1 in t and C^2 in x. Then*

$$h(t, Z_t) - h(0, Z_0)$$
$$= \int_0^t \dot{h}(s, Z_s)\, ds + \int_0^t h'(s, Z_s)\, dR_s + \int_0^t h'(s, Z_s)\, H_s\, dB_s + \tfrac{1}{2}\int_0^t h''(s, Z_s)\, H_s^2\, ds$$

6.5 Bessel processes

The Bessel process is the stochastic process that satisfies

$$dX_t^x = a\frac{dt}{X_t^x} + dB_t, \quad X_0^x = x \tag{235}$$

The superscript x indicates that we start the process at x. Throughout this section we consider only a real valued Bessel process. Note that if X_t satisfies the SDE, then so does $-X_t$. So we can restrict attention to $x > 0$. There is an implicit assumption here - that a solution exists,i.e., there is a stochastic process satisfying the above. If write the SDE as an integral equation then we can "shift" it back to the SLE integral equation and conclude that a solution exists up until the time when X_t goes to zero. Define

$$T_x = \inf\{t : X_t^x = 0\} \tag{236}$$

This is Lawler's definition. I don't know what it means. I think of T_x as the sup of the times for which the solution exists.

The Bessel process satisfies

$$X_t^x = x + B_t + a\int_0^t \frac{ds}{X_s^x} \tag{237}$$

Lemma 2 *If $0 < x < y$ then $0 < X_t^x < X_t^y$ for $t < T_x$. In particular, $0 < x < y$ implies $T_x \leq T_y$.*

Proof:

$$X_t^x = x + B_t + a \int_0^t \frac{ds}{X_s^x} \tag{238}$$

and

$$X_t^y = y + B_t + a \int_0^t \frac{ds}{X_s^y} \tag{239}$$

where we use the same B_t in both equations. Subtracting these two equations the B_t's cancel and we see that $X_t^y - X_t^x$ satisfies the differential equation

$$\frac{d}{dt}(X_t^y - X_t^x) = a \frac{X_t^x - X_t^y}{X_t^x X_t^y} \tag{240}$$

Note that this is negative, so X_t^x and X_t^y move closer together.

Suppose they collide, i.e., there is a T with $T < T_x$ and $T < T_y$ and $X_T^x = X_T^y$. We take T to be the first such time. Both X_t^x and X_t^y are continuous on $[0, T]$ and do not equal zero for these times, so they are bounded away from zero. So there is an $\epsilon > 0$ with $X_t^x, X_t^y \geq \epsilon$ for $t \leq T$. So

$$\frac{d}{dt}(X_t^y - X_t^x) \geq \frac{a}{\epsilon^2}(X_t^x - X_t^y) \tag{241}$$

and so

$$\frac{d}{dt} e^{at/\epsilon^2}(X_t^y - X_t^x) \geq 0 \tag{242}$$

Thus

$$X_t^y - X_t^x \geq e^{-at/\epsilon^2}(y - x) \tag{243}$$

for $t \leq T$, which contradicts the collision. ∎

The scaling of Brownian motion gives a scaling for the Bessel process.

Proposition 26 *Let X_t^x be the Bessel process and define $Y_t = x^{-1} X_{x^2 t}^x$. Then Y_t satisfies the Bessel SDE with $Y_0 = 1$.*

Proof: Consider the integral form of the Bessel equation for X_t. Doing the trivial substitution $t \to x^2 t$ and dividing the equation by x gives

$$x^{-1} X_{x^2 t}^x = 1 + x^{-1} B_{x^2 t} + \int_0^{x^2 t} x^{-1} \frac{ds}{X_s^x} \tag{244}$$

With a change of variables in the integral we have

$$Y_t = 1 + \hat{B}_t + \int_0^t \frac{ds}{x^{-1} X_{x^2 s}^x} \tag{245}$$

where $\hat{B}_t = x^{-1} B_{x^2 t}$. Note that $\hat{B}_t$ is a standard Brownian motion by the scaling of BM. So

$$Y_t = 1 + \hat{B}_t + \int_0^t \frac{ds}{Y_s} \tag{246}$$

i.e., Y_t satisfies the Bessel SDE. ∎

Lemma 3

$$P(T_x = \infty \quad and \quad \exists\, M\, s.t. \forall\, t \geq 0,\, X_t^x \leq M) = 0 \tag{247}$$

Corollary: Let $0 < x_1 < x < x_2$. Then

$$P(x_1 \leq X_t^x \leq x_2, \forall t \geq 0) = 0 \tag{248}$$

Proof: Obviously the event in the lemma does not change if we restrict M to the positive integers. This is a countable set, so it suffices to show that given an $M > 0$,

$$P(T_x = \infty \quad and \forall\, t \geq 0,\, X_t^x \leq M) = 0 \tag{249}$$

Consider a sample path B_t for which $T_x = \infty$ and $X_t^x \leq x_2$, for all $t \geq 0$. Then for all $t \geq 0$ we can rewrite the integral form of the Bessel SDE as

$$B_t = X_t^x - x - a \int_0^t \frac{ds}{X_s} \tag{250}$$

But using $X_s \leq x_2$ this implies

$$B_t \leq x_2 - x - \frac{at}{x_2} \tag{251}$$

The right side is a linear function of t which is negative for large t. So this implies there is a t_0 such that $B_t < 0$ for $t \geq t_0$. This event has probability zero. ∎

Proposition 27 *(i) If $a \geq \frac{1}{2}$, then with probability one, $T_x = \infty$ for $x \neq 0$.*
(ii) If $a = \frac{1}{2}$, then with probability one, $\inf_t X_t^x = 0$ for $x \neq 0$.
(iii) If $a > \frac{1}{2}$, then with probability one, $X_t^x \to \infty$ for $x \neq 0$.
(iv) If $a < \frac{1}{2}$, then with probability one, $T_x < \infty$ for $x \neq 0$.
(v) If $\frac{1}{4} < a < \frac{1}{2}$ and $0 < x < y$, then $P(T_x = T_y) > 0$.

Proof: We will only prove (i) and (iv). Let $h(x) = x^p$. Apply the Ito formula to $h(X_t^x)$. We get

$$d(X_t^x)^p = p(X_t^x)^{p-1}(\frac{a}{X_t^x}dt + dB_t) + \frac{p(p-1)}{2}(X_t^x)^{p-2}\, dt \tag{252}$$

The coef of dt is

$$p(X_t^x)^{p-2}[a+\frac{1}{2}(p-1)] \tag{253}$$

This vanishes if $p = 1 - 2a$. Thus $(X_t^x)^{1-2a}$ is a local martingale. Note that for the borderline case of $a = 1/2$ we have $p = 0$. For $a = 1/2$ we can show $\ln(X_t^x)$ is a local martingale.

Now let $x_1 < x < x_2$ and define a stopping time

$$\tau = \inf\{t : X_t^x = x_1 \quad or \quad x_2\} \tag{254}$$

The above lemma says that τ is finite with probability one. The optional sampling theorem says

$$E[(X_\tau^x)^p] = E[(X_0^x)^p] = x^p \tag{255}$$

Since

$$E[(X_\tau^x)^p] = x_1^p P(X_\tau^x = x_1) + x_2^p P(X_\tau^x = x_2) \tag{256}$$

Using $P(X_\tau^x = x_1) = 1 - P(X_\tau^x = x_2)$ we find

$$P(X_\tau^x = x_2) = \frac{x^p - x_1^p}{x_2^p - x_1^p} \tag{257}$$

for $a \neq 1/2$.

Now we prove (i). We assume that $a > 1/2$ and leave the case of $a = 1/2$ to the reader. So $p < 0$. Note that (257) then says that $P(X_\tau^x = x_2)$ converges to 1 as $x_1 \to 0^+$. For n with $1/n < x$, let E_n be the event that X_t^x hits x_2 before $1/n$. Then E_n is an increasing sequence of events whose probability converges to 1. So $P(\cup_n E_n)$ must be one. If the event $\cup_n E_n$ occurs then X_t^x reaches x_2. So $P(X_t^x\ reaches\ x_2) = 1$. This is true for any $x_2 > x$, in particular for all positive integers greater than x. So $P(X_t^x\ reaches\ k, \forall k) = 1$. Thus the probability that X_t^x is unbounded is one. If $T_x < \infty$ then X_t^x converges to 0 as $t \to T_x^-$ and from this we can see that X_t^x is bounded on $[0, T_x)$. So $P(T_x < \infty) = 1$.

The above was for a single x. It immediately implies $P(T_n < \infty, n = 1, 2, 3, \cdots) = 1$. Since $x < y$ implies $T_x \leq T_y$, this implies $P(T_x < \infty, \forall x > 0) = 1$.

Next we prove (iv), so $a < 1/2$ and thus $p > 0$. Since $x < y$ implies $T_x \leq T_y$, if T_y is finite then so is T_x for all $x < y$. So it suffice to prove $P(T_n < \infty, n = 1, 2, 3, \cdots) = 1$, and so it suffice to prove $P(T_n < \infty) = 1$ for $n = 1, 2, 3, ...$ By scaling this probability is the same for all n, so we will prove $P(T_1 < \infty) = 1$. We will write X_t^1 as just X_t.

Define for $n = 2, 3, \cdots$

$$E_n = \{\exists T < T_x\ with\ X_T = n\} \tag{258}$$

and

$$E_{n,k} = \{X_t\ reaches\ = n\ before\ 1/k\} \tag{259}$$

Then

$$P(E_{n,k}) = \frac{1^p - k^{-p}}{n^p - k^{-p}} \tag{260}$$

We claim $\cup_k E_{n,k} = E_n$. The $\subset$ inclusion is obvious. Suppose we are in $E_{n,k}$. So there is a T with $X_T = n$ X_t is a continuous function of $[0,T]$, so it attains its minimum and this minimum is not zero since $T < T_x$. So there is a k such that $X_t > 1/k$ for $t \leq T$. Thus we are in E_n. So $E_{n,k} \subset E_n$ for all k, proving the claim. As a sequence in k, $E_{n,k}$ is increasing, so

$$P(E_n) = \lim_k P(E_{n,k}) = 1/n^p \tag{261}$$

and so $P(E_n)$ goes to zero as $n \to \infty$. Since E_n is a decreasing sequence of events, $P(\cap_n E_n) = 0$. So with probability one, some E_n does not occur.So

$$P(\exists n \; such \; that \; X_t < n \; for \; t < T_x) = 1 \tag{262}$$

By the previous lemma, the probability that there is an n such that $X_t < n$ for $t < T_x$ **and** $T_1 = \infty$ is zero. So $P(T_1 < \infty)$ must be one. ∎