7 Properties of SLE

7.1 Phases

Recall the SLE equation:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z \tag{263}$$

Recall that

$$T_z = \sup\{t \geq 0 : g_s(z) \quad exists \quad 0 \leq s < t\} \tag{264}$$

and

$$H_t = \{z \in \mathbb{H} : T_z > t\}, \quad K_t = \{z \in \mathbb{H} : T_z \leq t\} = \mathbb{H} \setminus H_t \tag{265}$$

As we saw in the last section, if we let

$$Z_t = \frac{1}{\sqrt{\kappa}} g_t - B_t \tag{266}$$

this becomes

$$dZ_t = \frac{2}{\kappa}\frac{dt}{Z_t} - dB_t \tag{267}$$

This is the Bessel SDE **except** that Z_t is complex valued. However, for real intial conditions z, $g_t(z)$ remains real, and we do get exactly the Bessel process. Remembering that the parameter a in the Bessel process is related to κ by $a = \frac{2}{\kappa}$, we have

Proposition 28 *(i) If $\kappa \leq 4$ then with probability one, $T_x = \infty$ for all nonzero real x.*
(ii) If $\kappa > 4$ then with probability one, $T_x < \infty$ for all nonzero real x.
(iii) If $\kappa \geq 8$ then with probability one, $T_x < T_y$ for all real $x < y$.
(iv) If $4 < \kappa < 8$ and $x < y$ then $P(T_x = T_y) > 0$.

We will assume that K_t is generated by a curve $\gamma(t)$ which lies in the closed upper half plane, $\bar{\mathbb{H}}$. Recall that this means that for each t, the complement of K_t is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. This is a big assumption. The proof of this is hard.

Proposition 29 *If $\kappa \leq 4$ then with probability one, γ is a simple curve which does not intersect the real axis.*

Proof: For $s > 0$ we define

$$\gamma^s(t) = g_s(\gamma(t+s)) - \sqrt{\kappa}B_t \tag{268}$$

Then for each s, γ^s has the same distribution as γ.

Suppose that $\gamma(t_1) = \gamma(t_2)$ for some $0 \leq t_1 < t_2$. Pick a rational s in (t_1, t_2). Then $\gamma^s(t_2 - s)$ is on the real axis. So to show γ is simple it suffice to show that the probability that γ^s does not hit the real axis for all rational s is one. And so it suffices to show this for a single s. But γ^s has the same distribution as γ, so it suffices to show the probability γ never hits the real axis is one. If $\gamma(t) = x \in \mathbb{R}$, then $T_x < \infty$. But for $\kappa \leq 4$, we know the probability that $T_x = \infty$ for all nonzero real x is one. ∎

Proposition 30 *If $\kappa > 4$ then with probability one, $\cup_t K_t = \mathbb{H}$.*

Proof: Let $T = \max\{T_1, T_{-1}\}$. The point 1 must either lie on $\gamma[0, T_1]$ or it must be disconnected from ∞ by this curve. In the latter case the curve must hit the real axis to the right of 1. A similar statement applies to -1. So the curve hits the real axis on both sides of 0. It follows that there is an $\epsilon > 0$ such that $B_\epsilon(0) \cap \mathbb{H}$ is disconnected from ∞ by the curve and so is contained in K_T. Thus

$$P(\exists t, \epsilon \; B_\epsilon(0) \cap \mathbb{H} \subset K_t) = 1 \tag{269}$$

Thus

$$P(\cup_k \{\exists t : B_{1/k}(0) \cap \mathbb{H} \subset K_t\}) = 1 \tag{270}$$

So given $\delta > 0$, there is a k such that

$$P(\exists t : B_{1/k}(0) \cap \mathbb{H} \subset K_t) \geq 1 - \delta/2 \tag{271}$$

This probability equals

$$P(\cup_n \{B_{1/k}(0) \cap \mathbb{H} \subset K_n\}) \tag{272}$$

and so there is an n such that

$$P(B_{1/k}(0) \cap \mathbb{H} \subset K_n) \geq 1 - \delta \tag{273}$$

Now let $r > 0$. Then we have $B_{1/k}(0) \cap \mathbb{H} \subset K_n$ if and only if $B_r(0) \cap \mathbb{H} \subset rkK_n$. Recall that the scaling of SLE implies that rkK_n has the same distribution as $K_{r^2k^2n}$. Thus for all $\delta > 0, r > 0$ there exits k, n such that

$$P(B_r(0) \cap \mathbb{H} \subset K_{r^2k^2n}) \geq 1 - \delta \tag{274}$$

So for all $\delta > 0, r > 0$ there exits t such that

$$P(B_r(0) \cap \mathbb{H} \subset K_t) \geq 1 - \delta \tag{275}$$

So for all $r > 0$,

$$P(B_r(0) \cap \mathbb{H} \subset \cup_t K_t) = 1 \tag{276}$$

This proves the proposition. ∎

We now consider just how the curve γ generates K_t. Obviously, $\gamma[0, t] \subset K_t$. What can we say about the points in K_t which are not on the curve $\gamma[0, t]$?

Definition 24 *We say a point $z \in \mathbb{H}$ is "swallowed" if $T_z < \infty$ and*

$$z \notin \overline{\cup_{t<T_z} K_t} \tag{277}$$

Lemma 4 *If $T_z < \infty$, then either $z = \gamma(T_z)$ or z is swallowed.*

Proof: Suppose z is not swallowed so that

$$z \in \overline{\cup_{t<T_z} K_t} \tag{278}$$

Then there exists $t_n < T_z$ and $z_n \in K_{t_n}$ such that $z_n \to z$. We want to show that $z \in \gamma[0, T_z]$. For each n either z_n is on $\gamma[0, t_n]$ or it is not. In the latter case z_n and z are in different components of $\mathbb{H} \setminus \gamma[0, t_n]$, so the line segment from z_n to z must intersect this curve. So in both cases there is an $s_n \leq t_n$ such that $\gamma(s_n)$ is as close to z as z_n. Thus $\gamma(s_n)$ converges to z. The s_n are all bounded by T_z, so passing to a subsequence if necessary we can assume s_n converges, say to s. By the continuity of γ, $\gamma(s_n)$ converges to $\gamma(s)$. We already know it converges to z, so $z \in \gamma[0, T_z]$, which implies $z = \gamma(T_z)$. ∎

Lemma 5 *As an extended real-valued function, T_x is a continuous function of x.*

Proof: Since it is an increasing function, we only need to show it does not have any jump discontinuities. Suppose it has one at x. So there is a $T > 0$ and $\delta > 0$ such that $T_y \leq T - \delta$ if $y < x$ and $T_y \geq T - \delta$ if $y > x$. Let $y > x$. Then X_t^y exists and is nonzero on $[0, T]$. So there is a (random) $\epsilon > 0$ such that $X_t^y \geq \epsilon$ for $t \leq T$. Thus

$$y + B_t + a \int_0^t \frac{ds}{X_s^y} \geq \epsilon \tag{279}$$

But $X_s^x \leq X_s^y$, so this implies

$$y + B_t + a \int_0^t \frac{ds}{X_s^x} \geq \epsilon \tag{280}$$

and so $X_t^x \geq \epsilon - (y - x)$ for $t \leq T$. Choosing $y = x + \epsilon/2$, this implies $T_x \geq T$, a contradiction. ∎

Lemma 6 *Let $0 < a < b$. If $\gamma[0, \infty) \cap (a, b) = \emptyset$ then $T_a = T_b$.*

Proof: Let $x \in (a, b)$, so $x \notin \gamma[0, \infty)$. First consider the case that $T_b < \infty$. Then $T_x < \infty$. By a previous lemma, x must be swallowed. By the definition this implies there is a $\delta > 0$ such that $y \in (x - \delta, x + \delta)$ impies

$$y \notin \overline{\cup_{t<T_x} K_t} \tag{281}$$

In particular, $y \notin K_t$ for $t < T_x$.So $T_y \geq T_x$. For $y \in (x-\delta, x]$ this implies $T_y = T_x$. So T_y is constant on $(x-\delta, x]$. Take x to be the smallest x such that $T_x = T_b$ and this gives a contradiction.

Now suppose that $T_b = \infty$. We need to show $T_a = \infty$. If not, by the continuity of T_x, there is an $x \in (a, b)$ such that $T_x < \infty$ and $T_x > T_a$. Now apply the previous argument. ∎

Proposition 31 *If $\kappa \geq 8$, then γ is a space filling curve, i.e., $\gamma[0,\infty) = \overline{\mathbb{H}}$.*

Proof: We only prove that $\mathbb{R} \subset \gamma[0,\infty)$. Since $\kappa \geq 8$ we know that $x < y$ implies $T_x < T_y$ with probability one. By the previous lemma this implies that every interval not containing the origin intersects $\gamma[0,\infty)$. Since $\gamma[0,\infty) \cap \mathbb{R}$ is closed, it must be all of $\mathbb{R}$. ∎

Proposition 32 *For all κ, with probability one, $\lim_{t\to\infty} |\gamma(t)| = \infty$.*

The proof for κ is in Werner's St. Flour notes and for all κ in Lawler's book.

7.2 Some computations

There are many things about SLE that cannot be explicitly computed For example, for a fixed time t, $\gamma(t)$ is a (vector valued) random variable and one would like to know its distribution. This is not known. There are a few things that can be computed explicitly. We consider one such property here. We take $\kappa > 4$, so eventually K_t absorbs every point on the real axis. Let $a < 0 < c$. We will compute the probability that c is absorbed before a.

Another probability that may be computed explicitly is the following. Fix a point z in the upper half plane. The curve $\gamma(t)$ goes to infinity, so one can define whether it passes to the right or left of z. For any value of $\kappa > 0$ the probability it passes to the right is known in terms of a hypergeometric function.

Theorem 26 *Let $a < 0 < c$. Define*

$$E_{a,c} = \{K_t \ hits\ [c,\infty)\ before(-\infty, a]\} \tag{282}$$

Then

$$P(E_{a,c}) = F(\frac{-a}{c-a}) \tag{283}$$

where F is the solution of the ODE

$$\frac{\kappa}{4}F''(x) + (\frac{1}{x} - \frac{1}{1-x})F'(x) = 0 \tag{284}$$

with boundary conditions $F(0) = 0$ and $F(1) = 1$.

The proof can be found in section 3.2 of Werner's St. Flour lectures. It makes use of the following fact that is of general interest.

Lemma 7 *Let X be an integrable random variable. Define*

$$X_t = E(X|\mathcal{F}_t) \tag{285}$$

Then X_t is a martingale.

Proof: Let $t > s$. Then

$$E(X_t|\mathcal{F}_s) = E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s) = X_s \tag{286}$$

■

Using conformal invariance above theorem says something about SLE in other domains. Consider an equilateral triangle with vertices at the origin 0 and A and C. Let Φ be a conformal map of the upper half plane onto the triangle with $\Phi(a) = A, \Phi(b) = B, \Phi(0) = 0$. Let $U = \Phi(\infty)$. Note that since ∞ is "between" a and c, U belongs to the AC edge of the triangle. The map Φ takes SLE in the half plane from 0 to ∞ to SLE in the triangle from 0 to U. Let $\hat{\gamma}$ denote the SLE curve in the triangle. The theorem says that the probability it hits the segment CU before it hits the segment AU is $F(-a/(c-a))$. You can find Φ fairly explicitly. It is a Schwarz-Christoffel transformation. Thus you can find U. Amazingly, if $\kappa = 6$ it turns out that

$$F(-a/(c-a)) = \frac{|AU|}{|AC|} \tag{287}$$

where $|AU|$ just means the length of the segment. Recall that $\kappa = 6$ corresponds to the scaling limit of percolation. So the probability the percolation exploration process in the triangle from 0 to U hits the segment CU before AU is just the length of AU divided by the length of the triangles sides.

Cardy's formula is lurking here somehow - Lawler has much more on this. For certain other special values of κ there is an analogous result for other triangles due to Dubedat. See Werner for a discussion of this.

We now compute a different quantity. First suppose that $\kappa \leq 4$ so that the SLE curve γ is simple. Fix a point $z_0 = x_0 + iy_0$ in the upper half plane. The curve γ goes to ∞ and so we can define whether it passes to the right or left of the point z_0. This can be defined even when $4 < \kappa < 8$ using winding numbers as follows. Consider a large time T and let C_t be the curve which follows $\gamma[0,t]$ from 0 to $\gamma(T)$, then follows the arc of radius $|\gamma(T)|$ from $\gamma(T)$ to $|\gamma(T)|$ and finally follows the real axis from $|\gamma(T)|$ to 0. Then we say that γ passes to the left of z_0 if the winding number of C_t about z_0 is 1 for large t. We will compute the probability that γ passes to the left of z_0. This is in a paper by Schramm, "A percolation formula," arxiv:math.PR/0107096.

We begin the proof as follows. Let X_t and Y_t be real valued processes such that

$$g_t(z) - \sqrt{\kappa} B_t = X_t + i Y_t \tag{288}$$

Both X_t and Y_t depend on the "parameter" z, but we do not make this explicit.

Lemma 8 *The curve γ passes to the left of z if and only if*

$$\lim_{t \to T_z} W_t = \infty \tag{289}$$

where $\frac{W_t = X_t}{Y_t}$.

A careful proof may be found in Schramm's paper. A hueristic understanding is obtained by consider the flow of γ under g_t.

Theorem 27 *Let $0 < \kappa < 8$. Let $h(w)$ be the solution of*

$$\frac{\kappa}{2} h''(w) + \frac{4w}{w^2+1} h'(w) = 0 \tag{290}$$

with the boundary conditions $h(-\infty) = 0$, $h(\infty) = 1$, Then

$$P(\gamma \, passes \, left \, of \, x_0 + i y_0) = h(x_0/y_0) \tag{291}$$

The solution of the differential equation is a hypergeometric function. For $\kappa = 2, 8/3, 4$ and 8 the solution may be found in terms of elementary functions.

Proof: The SLE equation is

$$d(X_t + iY_t) = \frac{2}{X_t + iY_t} dt - \sqrt{\kappa} dB_t = \frac{2(X_t - iY_t)}{X_t^2 + Y_t^2} dt - \sqrt{\kappa} dB_t \tag{292}$$

So

$$dX_t = \frac{2X_t}{X_t^2 + Y_t^2} dt - \sqrt{\kappa} dB_t \tag{293}$$

$$dY_t = \frac{-2Y_t}{X_t^2 + Y_t^2} dt \tag{294}$$

Note that $(dX_t)^2 = \kappa dt$ and $(dY_t)^2 = 0$. So Ito's lemma and some algebra yields

$$dW_t = \frac{1}{Y_t} dX_t - \frac{X_t}{Y_t^2} dY_t + \frac{1}{2} \frac{2X_t}{Y_t^3} (dY_t)^2 = \frac{4W_t}{X_t^2 + Y_t^2} dt - \sqrt{\kappa} dB_t \tag{295}$$

We make a time change. Let

$$s(t) = \int_0^t \frac{du}{Y_u^2} \tag{296}$$

and let $r(s)$ be its inverse, so

$$\int_0^{r(s)} \frac{du}{Y_u^2} = s \tag{297}$$

Define

$$\hat{B}_s = \int_0^{r(s)} \frac{dB_u}{Y_u} \tag{298}$$

Then

$$< \hat{B} >_s = \int_0^{r(s)} \frac{dB_u}{Y_u^2} = s \tag{299}$$

So $\hat{B}_s$ is a standard Brownian motion with respect to the filtration $\hat{\mathcal{F}}_s = \mathcal{F}_{r(s)}$. We define $\hat{W}_s = W_{r(s)}$. Then the SDE for W_t becommes

$$d\hat{W}_s = \frac{4\hat{W}_s}{\hat{W}_s + 1} ds - \sqrt{\kappa} d\hat{B}_s \tag{300}$$

Now let $a < b$ and for $w \in [a, b]$ define $h(w)$ to be the probabilty that W_t hits b before a if W_t starts at w (Obviously h depends on a and b, but we will not make this explicit.) Now consider the process $h(W_t)$. We claim it is a martingale. To see this consider $P(E_{a,b}|\hat{\mathcal{F}}_s)$ where $E_{a,b}$ is the event that $\hat{W}_t$ hits b before a. If we know $\hat{\mathcal{F}}_s$, then we know $\hat{W}_s$, and what $\hat{W}_t$ does after time s is independent of $\hat{\mathcal{F}}_s$. Thus

$$P(E_{a,b}|\hat{\mathcal{F}}_s) = h(\hat{W}_t) \tag{301}$$

and so $h(\hat{W}_u)$ is a martingale.

Now we use Ito to compute $dh(\hat{W}_s$.

$$dh(\hat{W}_s) = h'(\hat{W}_s) d\hat{W}_s + \frac{1}{2} h''(\hat{W}_s)(d\hat{W}_s)^2 \tag{302}$$

Using the above computation of $d\hat{W}_s$ this implies the coeffecient of ds in $dh(\hat{W}_s)$ is

$$\frac{\kappa}{2} h''(\hat{W}_s) + \frac{4\hat{W}_s}{\hat{W}_s^2 + 1} h'(\hat{W}_s) \tag{303}$$

This must be zero for $h(\hat{W}_s)$ to be a martingale. So h satisisfies the differential equation of the theorem. The boundary conditions are $h(a) = 0,\ h(b) = 1$. We send a to $-\infty$ and b to ∞ to complete the proof. ∎

7.3 Locality property

We briefly recall the locality property in the context of the percolation process. Consider percolation in the upper half plane and fix the hexagons to the left of the origin to be white and those right of the origin to be grey so that there will be an interface starting at the origin. Following this interface is a local process- at each vertex you look at the three hexagons around you and use them to determine whether to go left or right. The hexagons are colored independently, so we conclude that if we were to perturb the half plane in some region away from the origin, the exploration process would be the same up to the time it enters this perturbed region. Let A be a hull and let $\mathbb{H} \setminus A$ be the perturbed half plane. So the exploration processes in $\mathbb{H}$ and $\mathbb{H} \setminus A$ are the same up to the time they enter A. Now let Φ be the conformal map of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with the usual hydrodynamic normalization.

Fix a hull A which is a positive distance from the origin. In this section all conformal maps will be normalized with the hydrodynamic normalization. Let Φ be the conformal map of $\mathbb{H} \setminus A$ onto $\mathbb{H}$.

Let K_t be an SLE process in the half plane. Let $g_t(z)$ be the associated conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$. Define

$$\tilde{K}_t = \Phi(K_t) \tag{304}$$

Let $\tilde{g}_t$ be the conformal map of $\mathbb{H} \setminus \tilde{K}_t$ onto $\mathbb{H}$. The composition $\tilde{g}_t \circ \Phi$ maps $\mathbb{H} \setminus (K_t \cup A)$ onto $\mathbb{H}$. Note that we first "removed" A and then $\tilde{K}_t$. We can remove them in the other order. g_t maps $\mathbb{H} \setminus (K_t \cup A)$ onto $\mathbb{H} \setminus g_t(A)$. Then we define h_t to be the conformal map from $\mathbb{H} \setminus g_t(A)$ onto $\mathbb{H}$. Thus $h_t \circ g_t$ maps $\mathbb{H} \setminus (K_t \cup A)$ onto $\mathbb{H}$. There is only one conformal map that does this with the hydrodynamic normalization, so we must have

$$h_t \circ g_t = \tilde{g}_t \circ \Phi \tag{305}$$

The process $\hat{K}_t$ satisfies the conditions to come from Lowener's equation. However, there is no reason the half-plane capacity of $\hat{K}_t$ should be a linear function of t. So the process $\hat{K}_t$ is a generalized Loewner chain. Define $a(t) = A(A \cup K_t)$. The picture shows this equals $a(A) + a(\tilde{K}_t)$. So $\tilde{g}_t$ satifes the generalized Loewner equation:

$$\partial_t \tilde{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - \tilde{W}_t} \tag{306}$$

We need to understand $\tilde{W}_t$ and $\dot{a}(t)$. We will show there is a random time change which changes $\tilde{W}_t$ into $\sqrt{\kappa}$ times a Brownian motion and which also changes $\dot{a}(t)$ into 2. Thus under this time change $\tilde{K}_t$ becomes SLE.

Lemma 9

$$\dot{a}(t) = 2[h_t'(W_t)]^2 \tag{307}$$

Proof: We give the idea. For a careful proof see Lawler's book. For $\epsilon > 0$, let $\Delta_\epsilon = K_{t+\epsilon} \setminus K_t$. Then Δ_ϵ is a small set near $\gamma(t)$. So $g_t(\Delta_\epsilon)$ is a small set near $g_t(\gamma(t)) = W_t$. The Markov property of SLE says that $g_t(\Delta_\epsilon)$ is another SLE if we shift it appropriately, and it has the correct time parameterization. So $a(g_t(\Delta_\epsilon) = 2\epsilon$. By Schwarz reflection, the map h_t has an analytic continuation to a neighborhood of W_t. So near W_t, h_t acts like multiplication by $h_t'(W_t)$. Note that this number is real. The scaling property of capacity is $a(\lambda K) = \lambda^2 a(K)$. So

$$a(h_t(g_t(\Delta_\epsilon))) \approx [h_t'(W_t)]^2 2\epsilon \tag{308}$$

Now

$$a(K_{t+\epsilon} \cup A) = a(K_t \cup A) + a(h_t(g_t(\Delta_\epsilon))) \tag{309}$$

The lemma follows. ■

We know that $g_t(\gamma(t)) = W_t$. So $h_t(g_t(\gamma(t))) = h_t(W_t)$. By (305) this equals

$$\tilde{g}_t(\Phi(\gamma(t))) = \tilde{g}_t(\tilde{\gamma}(t)) = \tilde{W}_t \tag{310}$$

Thus

$$\tilde{W}_t = h_t(W_t) \tag{311}$$

From (305) we have

$$h_t = \tilde{g}_t \circ \Phi \circ g_t^{-1} \tag{312}$$

We want to differentiate this with respect to t. In doing this calculation we should keep in mind that $g_t(z)$ is a function of two variables and the compositions above are in the z variable. It is perhaps better to write $g_t(z)$ as $g(t,z)$, and the above equation as

$$h(t,z) = \tilde{g}(t, \Phi(g^{-1}(t,z))) \tag{313}$$

From the SLE equation we find

$$\frac{\partial g^{-1}}{\partial t}(t,z) = \frac{-2(g^{-1})'(t,z)}{z - W_t} \tag{314}$$

and we eventually find

$$\dot{h}_t(z) = \frac{2h_t'(W_t)^2}{h_t(z) - \tilde{W}_t} - \frac{2h_t'(z)}{z - W_t} \tag{315}$$

This holds for $z \in \mathbb{H} \setminus g_t(A)$. We can take the limit as $z \to W_t$ since h_t is analytic in neighborhood of W_t. After some calculus we find

$$\dot{h}_t(W_t) = -3h''(W_t) \tag{316}$$

Now we use Ito to compute $d\,\tilde{W}_t = d\,h_t(W)$.

$$d\,h_t(W) = \dot{h}_t(W_t)\,dt + h_t'(W_t)\,dW_t + \frac{1}{2}h_t''(W_t)\,(dW_t)^2 \tag{317}$$

Using $W_t = \sqrt{\kappa} B_t$, we have $(dW_t)^2 = \kappa dt$ and so

$$d\tilde{W}_t = (\frac{\kappa}{2} - 3) h_t''(W_t)\, dt + \sqrt{\kappa} h_t'(W_t)\, dB_t \tag{318}$$

Thus if $\kappa = 6$ the dt term drops out and

$$d\tilde{W}_t = \sqrt{\kappa} h_t'(W_t)\, dB_t \tag{319}$$

Define $\tilde{B}_t = \tilde{W}_t / \sqrt{\kappa}$, so

$$d\tilde{B}_t = h'(W_t) dB_t \tag{320}$$

This implies that there is random time change which will make $\tilde{B}_t$ into a standard Brownian motion. To find the time change we compute the quadratic variation of $\tilde{B}_t$:

$$< \tilde{B} >_t = \int_0^t h'(W_u)^2\, du = \int_0^t \frac{1}{2} \dot{a}(u)\, du = \frac{a(t)}{2} \tag{321}$$

Let $t(s)$ be the inverse function to $a(t)/2$. So

$$\int_0^{t(s)} h'(W_u)^2\, du = s \tag{322}$$

Note that

$$\frac{ds}{dt} = h'(W_t)^2 = \dot{a}(t) \tag{323}$$

Define

$$\hat{B}_s = \tilde{B}_{t(s)} \tag{324}$$

Then $\hat{B}_s$ is a standard Brownian motion. Now define

$$\hat{g}_s = \tilde{g}_{t(s)} \tag{325}$$

Then

$$\partial_s \, \hat{g}_s = \dot{g}_{t(s)} \frac{dt}{ds} = \frac{\dot{a}(t(s))}{g_{t(s)} - \tilde{W}_{t(s)}} \left(\frac{ds}{dt} \right)^{-1} = \frac{2}{\hat{g}_s - \hat{W}_s} \tag{326}$$

Thus $\hat{g}_s$ is another SLE. We have proved the following.

Theorem 28 *Let $\kappa = 6$. Let A be a hull at a positive distance from the origin. Let T be the first time when K_t hits A. Let Φ be the conformal map of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with the hydrodynamic normalization. Then for $t < T$, $\Phi(K_t)$ is another SLE_6 in $\mathbb{H}$ from 0 to ∞.*

Splitting property

7.4 Restriction property

The locality property is easily confused with the restriction property when you first see them, but they are in fact quite different. Consider the SAW in the upper half plane. The measure is the uniform measure on walks in the upper half plane that do not intersect themselves. Fix a hull A. (Recall that this means that $\mathbb{H} \setminus A$ is simply connected.) If we condition on the event that the SAW does not hit A, then we will just get the uniform measure on SAW's in $\mathbb{H} \setminus A$, i.e., we get the SAW in $\mathbb{H} \setminus A$. This is the restriction property. Note that it involves a conditional probability while the locality property does not.

Theorem 29 *Consider chordal $SLE_{8/3}$ in $\mathbb{H}$. Then for any hull A,*

$$P(\gamma \cap A = \emptyset) = [\psi_A(0)']^{5/8} \tag{327}$$

We continue to use the notation from the previous section. In particular, W_t is $\sqrt{\kappa} B_t$, the driving function for the SLE $\gamma(t)$. The proof uses the stochastic process $h_t'(W_t)$. The two key facts about it are the following two lemmas.

Lemma 10 *The process $h_t'(W_t)^a$ is a local martingale if $\kappa = 8/3$ and $a = 5/8$.*

Lemma 11 *Let T be the first time when $\gamma(t) \in A_0$ with T defined to be ∞ if $\gamma \cap A_0 = \emptyset$. Consider*

$$\lim_{t \to T^-} h_t'(W_t) \tag{328}$$

If $T = \infty$ then the limit is 1 and if $T < \infty$ then the limit is 0.

We will use the the two lemmas to prove the theorem. Then we will prove the two lemmas.

Proof of theorem: h_t is a normalized map from a subset of $\mathbb{H}$ onto $\mathbb{H}$. Hence $|h_t'(z)| \leq 1$ and so $h_t'(W_t)^a$ is a bounded martingale. Since T is a stopping time,

$$E[h_T'(W_T)^{5/8}] = E[h_0'(W_0)^{5/8}] = \Phi'(0)^{5/8} \tag{329}$$

where we have used $h_0 = \Phi$. But lemma 11 says

$$E[h_T'(W_T)^{5/8}] = P[T = \infty] = P[\gamma \cap A_0 = \emptyset] \tag{330}$$

■

Proof of lemma 10: For $t < T$ we can differentiate (315) to get

$$\partial_t h'(z) = \frac{-2h_t'(W_t)^2 h_t'(z)}{(h_t(z) - \tilde{W}_t)^2} + \frac{2h_t'(z)}{(z - W_t)^2} - \frac{2h_t''(z)}{z - W_t} \tag{331}$$

Since h^t is analytic in a neighbor of W_t we can let $z \to W_t$ and after a lot of calculus we find

$$\partial_t h'(W_t) = \frac{h_t''(W_t)^2}{2h_t'(W_t)} - \frac{4}{3}h_t'''(W_t) \tag{332}$$

Now we use Ito to compute $d[h_t'(W_t)^a]$:

$$\begin{aligned} d[h_t'(W_t)^a] &= a h_t'(W_t)^{a-1}\partial_t h_t'(W_t)\, dt + a h_t'(W_t)^{a-1} h_t''(W_t)\, dW_t \\ &+ \left[\tfrac{a(a-1)}{2} h_t'(W_t)^{a-2}[h_t''(W_t)]^2 + \tfrac{a}{2} h_t'(W_t)^{a-1} h_t'''(W_t)\right] (dW_t)^2 \end{aligned} \tag{333}$$

Since $(dW_t)^2 = \kappa dt$, the coeffecient of dt is

$$a h_t'(W_t)^{a-1}\partial_t h_t'(W_t) + \frac{\kappa a(a-1)}{2} h_t'(W_t)^{a-2}[h_t''(W_t)]^2 + \frac{\kappa a}{2} h_t'(W_t)^{a-1} h_t'''(W_t) \tag{334}$$

Using (332) this becomes

$$\begin{aligned} & \tfrac{a}{2} h_t'(W_t)^{a-2}[h_t''(W_t)]^2 - a h_t'(W_t)^{a-1} \tfrac{4}{3} h_t'''(W_t) \\ & + \tfrac{\kappa a(a-1)}{2} h_t'(W_t)^{a-2}[h_t''(W_t)]^2 + \tfrac{\kappa a}{2} h_t'(W_t)^{a-1} h_t'''(W_t) \\ = \; & h_t'(W_t)^{a-2}\left[(\tfrac{a}{2} + \tfrac{\kappa a(a-1)}{2})[h_t''(W_t)]^2 + (\tfrac{\kappa a}{2} - \tfrac{4a}{3}) h_t'(W_t) h_t'''(W_t)\right] \end{aligned} \tag{335}$$

This vanishes if $\kappa = 8/3$ and $a = 5/8$. ∎

Idea of Proof of lemma 11: pictures and handwaving. See Lawler for an honest proof. ∎

We now use the previous theorem to prove the restriction property. Let A_0 be a hull whose distance from the origin is nonzero. Let ψ_{A_0} be the conformal map from $\mathbb{H} \setminus A_0$ onto $\mathbb{H}$. Then if γ is a curve in $\mathbb{H}$, $\psi_{A_0}^{-1}(\gamma)$ is a curve in $\mathbb{H}$ that does not intersect A_0. The following corollary gives its distribution in terms of an SLE in $\mathbb{H}$.

Corollary (restriction property): Consider chordal $\text{SLE}_{8/3}$ in $\mathbb{H}$. Let A_0 be a hull whose distance from the origin is nonzero. Let γ be $\gamma[0, \infty)$. The conditional distribution of γ given $\gamma \cap A_0$ is the same as the distribution of $\psi_{A_0}^{-1}(\gamma)$. The map ψ_{A_0} is the conformal map from $\mathbb{H} \setminus A_0$ onto $\mathbb{H}$.

Proof: To keep the notation under control we will denote ψ_{A_0} by just ψ_0. We show that for every hull A in $\mathbb{H}$ at a positive distance from 0 which is disjoint from A_0 we have

$$P(\gamma \cap A = \emptyset | \gamma \cap A_0 = \emptyset) = P(\psi_0^{-1}(\gamma) \cap A = \emptyset) \tag{336}$$

The definition of conditional probability says

$$P(\gamma \cap A = \emptyset | \gamma \cap A_0 = \emptyset) = P(\gamma \cap A = \emptyset, \gamma \cap A_0 = \emptyset)/P(\gamma \cap A_0 = \emptyset) \tag{337}$$

By the theorem the denominator is $\psi_0'(0)^{5/8}$. Now we compute the numerator.

$$P(\gamma \cap A = \emptyset, \gamma \cap A_0 = \emptyset) = P(\gamma \cap (A \cup A_0) = \emptyset) = \psi_{A \cup A_0}'(0)^{5/8} \tag{338}$$

We can construct the map $\psi_{A \cup A_0}$ in two steps. First we use the map ψ_0. This maps $\mathbb{H} \setminus (A \cup A_0)$ onto $\mathbb{H} \setminus \psi_0(A)$. Then we apply the map $\psi_{\psi_0(A)}$. This maps us onto $\mathbb{H}$. Thus $\psi_{A \cup A_0} = \psi_{\psi_0(A)} \circ \psi_0$. So

$$\psi_{A \cup A_0}'(0) = \psi_{\psi_0(A)}'(\psi_0(0))\psi_0'(0) = \psi_{\psi_0(A)}'(0)\psi_0'(0) \tag{339}$$

Thus

$$\begin{aligned} P(\gamma \cap A = \emptyset | \gamma \cap A_0 = \emptyset) &= [\psi_{\psi_0(A)}'(0)\psi_0'(0)]^{5/8}/\psi_0'(0)^{5/8} = \psi_{\psi_0(A)}'(0)^{5/8} \\ &= P(\gamma \cap \psi_0(A) = \emptyset) = P(\psi_0^{-1}(\gamma) \cap A = \emptyset) \end{aligned} \tag{340}$$

■

For some particular sets A one can find the map ψ_A explicitly, and so the probability the $\text{SLE}_{8/3}$ curve hits A can be explicitly computed. This gives all sorts of explicit predictions for the self-avoiding walk. We give two examples. Let X be the distance from the curve γ to the point 1 on the real axis. So X takes values in $(0, 1]$. For the second random variable we consider the intersections of γ with the vertical line $x = 1$. We define Y to be the y-coordinate of the lowest intersection. So Y takes values in $(0, \infty)$.

Consider the random variable X. For $a < 1$, let $A_a = \{z \in \mathbb{H} : |z - 1| \leq a\}$ The distance X is less than or equal to a if and only if γ hits A_a. So this probability can be computed if we can find the conformal map that sends $\mathbb{H} \setminus A_a$ onto $\mathbb{H}$. The map $\phi(z) = z + 1/z$ maps $\{z \in \mathbb{H} : |z| > 1\}$ onto $\mathbb{H}$. Some scaling and translating gives the needed map.

Exercise : Show that

$$P(X \leq t) = 1 - (1 - t^2)^{5/8}. \tag{341}$$

Note: be sure to normalize the conformal map correctly.

Now we find the distribution of Y. Let L_a to be the vertical line segment from 1 to $1 + ia$. The random variable Y is less than or equal to a if and only if the curve hits L_a. The conformal map that takes $H \setminus L_a$ onto H can be constructed using square roots and squares.

Exercise : Show that

$$P(Y \leq t) = 1 - (1 + t^2)^{-5/16} \tag{342}$$

7.5 Discrete SLE

In this section we consider the problem of approximating SLE by a "discrete SLE." The term discrete SLE is a bit misleading. The random process we will define will produce continuous curves in the upper half plane. The plane is not replaced by a lattice. The time interval is divided into small intervals of width δ and the times $n\delta$ will play a special role, but the random curves will still be defined for all time. The discrete SLE should converge to SLE as $\delta \to 0$. We can think of our discrete approximation as the result of replacing the Brownian motion in the driving function by some stochastic process that approximates Brownian motion and converges to it as $\delta \to 0$. For example, we could use an ordinary random walk with time steps of size δ and spatial steps chosen to make the variance of the random walk at time t equal to κt.

We begin by reviewing some facts about the Lowener equation,

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t} \tag{343}$$

Let $t, s > 0$. The map g_{t+s} maps $\mathbb{H} \setminus K_{t+s}$ onto $\mathbb{H}$. We can do this in two stages. We first apply the map g_s. This maps $\mathbb{H} \setminus K_s$ onto $\mathbb{H}$, but it maps $\mathbb{H} \setminus K_{t+s}$ onto $\mathbb{H} \setminus g_s(K_{t+s} - K_s)$. Let $\hat{g}_t$ be the conformal map that maps $\mathbb{H} \setminus g_s(K_{t+s} - K_s)$ onto $\mathbb{H}$ with the usual hydrodynamic normalization. So

$$g_{s+t} = \hat{g}_t \circ g_s, \quad i.e., \quad \hat{g}_t = g_{s+t} \circ g_s^{-1} \tag{344}$$

Then

$$\frac{d}{dt}\hat{g}_t(z) = \frac{d}{dt} g_{s+t} \circ g_s^{-1}(z) = \frac{2}{g_{s+t} \circ g_s^{-1}(z) - U_{s+t}} = \frac{2}{\hat{g}_t(z) - U_{s+t}} \tag{345}$$

Note that $\hat{g}_0(z) = z$. Thus $\hat{g}_t(z)$ is obtained by solving the Lowener equation with driving function $\hat{U}_t = U_{s+t}$. Note that this driving function starts at U_s and so the $\hat{K}_t$ associated with $\hat{g}_t$ starting growing at 0.

Let $\delta > 0$ be a small time interval. We first define the driving function U_t for t of the form $k\delta$ for integer k. Restricted to these times it will just be an ordinary random walk. Let X_i be an i.i.d. sequence with $X_i = \pm 1$ with equal probability. Then we define

$$U_{k\delta} = \sqrt{\kappa\delta} \sum_{i=1}^{k} X_i \tag{346}$$

Note that the variance of $U_{k\delta}$ is $k\kappa\delta$, the same as the variance of $\sqrt{\kappa} B_{k\delta}$, the driving function of SLE.

We postpone the definition of U_t in between the times $k\delta$ and for the moment just assume it is defined there so that U_t is a continuous function. Let

$$G_k = g_{k\delta} \circ g_{(k-1)\delta}^{-1} \tag{347}$$

So

$$g_{k\delta} = G_k \circ G_{k-1} \circ G_{k-2} \circ \cdots \cdots G_2 \circ G_1 \tag{348}$$

By the remarks above, $G_k(z)$ is obtained by solving the Loewner equation with driving function $U_{(k-1)\delta+t}$ for $t = 0$ to $t = \delta$. Note that G_k^{-1} maps $\mathbb{H}$ to $\mathbb{H}$ minus a cut or blob "centered" around $U_{(k-1)\delta}$. If we consider $G_k - U_{(k-1)\delta}$, it is obtained by solving the Loewner equation with driving function $U_{(k-1)\delta+t} - U_{(k-1)\delta}$ for $t = 0$ to $t = \delta$. Note that this driving function starts at 0 and ends at $\pm\sqrt{\kappa\delta}$. Let v_t be a function on $[0, \delta]$ with $v_0 = 0$ and $v_\delta = \sqrt{\kappa\delta}$. We will take

$$U_{(k-1)\delta+t} = U_{(k-1)\delta} \pm v_t \tag{349}$$

where the choice of $\pm$ is given by the sign of the X_k, i.e., the direction of the kth step of the random walk. The function v_t does not depend on k.

The key idea is to choose v_t so that the Loewner equation

$$\dot{G}_t(z) = \frac{2}{G_t(z) - v_t} \tag{350}$$

for $0 \le t \le \delta$ may be explicitly solved and the resulting G_δ is relatively simple. Actually, it is easier to start with a relatively simple G_δ that you would like to get and from it figure out what is v_t. We will assume that G_δ corresponds to a curve, i.e., $G_\delta^{-1}(\mathbb{H})$ is $\mathbb{H}$ minus a curve. There are two constraints on G_δ. The curve must have capacity 2δ and G_δ must map the tip of the curve to $v_\delta = \sqrt{\kappa\delta}$.

The map g_t maps the half plane minus the curve onto the half plane. Let f_t be the inverse map which sends the half plane onto the half plane minus the curve. In particular, $f_t(U_t) = \gamma(t)$. We compute an approximate curve by computing $f_t(U_t)$ at the times $t = k\delta$.

We will work out the details when G_δ is the following. Let C be a line segment starting at the origin with a polar angle of $\alpha\pi$. G_δ will map $\mathbb{H} \setminus C$ onto $\mathbb{H}$. There are two degrees of freedom for the line segment - its length and α. There are two constraints - the line segment must have capacity 2δ and the tip of the segment must get mapped to $\sqrt{\kappa\delta}$. Consider the segment whose capacity is 2 and whose tip gets mapped to $\sqrt{\kappa}$. If we scale it by a factor of $\sqrt{\delta}$ then the new segment has capacity 2δ and its tip is mapped to $\sqrt{\kappa\delta}$. Thus α depends only on κ, not on δ.

Consider the map

$$\phi(z) = (z+y)^{1-\alpha}(z-x)^\alpha \tag{351}$$

where $x, y > 0$. It maps the half plane onto the half plane minus a line segment which starts at the origin and forms an angle α with the positive real axis. The interval $[-y, x]$ gets mapped onto the slit. The length of this interval determines the length of the slit. Shifting this interval (relative to 0) does not change this length of the slit. We must position the interval (relative to 0) so that the hydrodynamic normalization is satisfied. This is essential for the Loewner eq. This is achieved by taking $x/y = (1-\alpha)/\alpha$.

We define

$$f_t(z) = \left(z + 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \left(z - 2\sqrt{t}\sqrt{\frac{1-\alpha}{\alpha}}\right)^{\alpha} \tag{352}$$

And then let $g_t(z) = f_t^{-1}(z)$. Some tedious but straightforward calculation shows $g_t(z)$ has capacity $2t$. So we know from the general theory that $g_t(z)$ must satisfy the Loewner equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t} \tag{353}$$

for some driving function U_t. Some calculation then shows that

$$U_t = c_\alpha \sqrt{t} \tag{354}$$

where

$$c_\alpha = 2\frac{1-2\alpha}{\sqrt{\alpha(1-\alpha)}} \tag{355}$$

Note that 0 is not the preimage of the tip. To find it, we maximize the distance from the origin along the slit. For values of x such that $f(x)$ is on the slit this length is

$$f_t(z) = \left(x + 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \left(-x + 2\sqrt{t}\sqrt{\frac{1-\alpha}{\alpha}}\right)^{\alpha} \tag{356}$$

Setting the derivative equals to zero leads to

$$x = \frac{2\sqrt{t}(1-2\alpha)}{\sqrt{\alpha(1-\alpha)}} = c_\alpha \sqrt{t} \tag{357}$$

Plugging this into f we find the length of the slit is

$$L = \frac{2\sqrt{t}}{\sqrt{\alpha(1-\alpha)}} \, (1-\alpha)^{1-\alpha} \, \alpha^\alpha \tag{358}$$

The constraint that the tip of the slit must be mapped to v_δ leads to $c_\alpha^2 = \kappa$. We can also see this as follows. Think of U_t as an approximation to $\sqrt{\kappa} B_t$ over a small time interval dt. We have $U_{dt} = \pm c_\alpha dt$ and we want the variance of U_{dt} to be $\kappa \, dt$. So

$$c_\alpha^2 = \kappa \tag{359}$$

So

$$\kappa = 4\frac{(1-2\alpha)^2}{\alpha(1-\alpha)} \tag{360}$$

which gives

$$16\alpha^2 + \kappa\alpha^2 - 16\alpha - \kappa\alpha + 4 = 0 \tag{361}$$

and so

$$\alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{\kappa}{16+\kappa}} \tag{362}$$

We take the solution with $\alpha < 1/2$.

The above discretization is natural in that it uses line segments to approximate the SLE path (in some sense). There is another approximation studied by R. Bauer that results in an approximation that is even easier to compute, although it has the strange feature that the driving function is not continuous. We continue to let

$$U_{k\delta} = \sqrt{\kappa\delta}\sum_{i=1}^{k} X_i \tag{363}$$

but now we take U_t to be constant in between these times. The constant is chosen so that the function is left continuous. (I think; it could be right cont.) If we take $v_t = 0$, then the Loewner equation just gives a vertical slit starting at 0.