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0 Introduction

> Mathematical physics is the scientific discipline concerned with the interface of mathematics and physics. There is no real consensus about what does or does not constitute mathematical physics.
>
> from the Wikipedia entry for "Mathematical Physics"

For centuries there has been an extremely fruitful interaction between mathematics and physics. Here are some examples:

Quantum mechanics $\iff$ linear operators on Hilbert spaces

General relativity $\iff$ Riemannian geometry

Classical statistical physics $\iff$ probability

String theory $\iff$ geometry

The fruitful interaction between physics and mathematics goes both ways. Theoretical physics often requires sophisticated mathematics and in fact physicists are often are involved in the development of the needed mathematics. Going the other way, physics provides a wealth of interesting mathematical models, conjectures about these models, and even ideas for proving the conjectures.

I agree with the Wikipedia quote, but I would like to offer an opinion on what mathematical physics should not be. It should not be a quest to put all of theoretical physics on a rigorous footing. Theoretical physics is not mathematics and should not be judged by the standards of mathematics. Complaining that an argument in theoretical physics is not mathematically rigorous is no more appropriate than complaining that a mathematical theorem is not valid because there is no physical experiment to confirm it.

This course is about the flow of models and ideas from physics, specifically classical statistical mechanics, to mathematics. The goal of this course is to introduce some of the important models, conjectures and theorems that have come from classical statistical mechanics. Roughly speaking, classical statistical mechanics studies the behavior of systems with a huge number of degrees of freedom, e.g., the molecules in a gas in some container. The classical refers to the fact that one treats the particles using classical mechanics.

(There is also quantum statistical mechanics.) Rather than attempt to study the individual behaviors of the huge number of particles, spins or whatever in the system, one attempts to can something about averages involving the degrees of freedom. For example, what is the average velocity of a molecule in the gas - hence the connection with probability.

Our main themes will be

Models In the example of the molecules in a gas the particles move about in $\mathbb{R}^3$ or some subset thereof. There are also models in which the degrees of freedom are tied to a lattice. For a crystal structure the degrees of freedom could be the spin of an electron at each site in the crystal. This course will focus almost entirely on such lattice models. The first models we will study are the Ising model (the prototypcial model for magnetic phase transitions) and percolation (originally introduced to model fluid flow in porous materials). These models have a discrete variable associated with each site. Next we consider various random walk models including self-avoiding walks (which are meant to model polymers in solution). Then we consider models like the Ising model in which there is a degree of freedom at each site in the lattice, but now the variable at each site will be continuous with some symmetry. Another ground of models of this type are Euclidean field theories in which the variable at each site takes values in the reals or complexs or $\mathbb{R}^n$.

Phase transitions: Physically, all of these models exhibit phase transitions in which an infinitesimal change in a parameter (e.g., temperature) produces a qualitative change in the system. A more probabilistic description of a phase transition is that typically the randomness which is in the model at microscopic scales does not manifest itself at macroscopic scales. But at a phase transition the microscopic randomness producess random structures at a macroscopic length scale.

Universality An important idea from physics is "universality" which says that many features of these phase transitions do not depend on the microscopic details of the model but only on some qualitative features of the model.

Renormalization Group Physicists's understanding of this universality is based on a set of ideas known as the renormalization group (RG). We will study this set of ideas in several concrete settings.

Misc If time permits we will also study how one does Monte Carlo simulations of these models. And if we discover that the semester is actually

twenty weeks long rather fourteen weeks, we will look at rigorous expansion technique (polymer expansions).

Tentative Outline

1. The Ising model (a lattice models with discrete variables)
   a. mathematical characterization
   b. phase transitions - entropy vs energy - Peierls argument
   c. universality
2. Percolation (another discrete lattice model)
3. Renormalization group for the Ising model
4. Random walk models
   a. Ordinary random walk, Brownian motion and the invariance principle
   b. A renormalization group approach to the central limit theorem
   c. Self avoiding random walks
   d. A renormalization group approach to SAW
5. Lattice models with continuous variables : XY and Heisenberg models
   a. role of symmetry in phase transitions - Mermin Wagner theorem
   b. effect of symmetry on critical phenomena
6. Euclidean field theory as classical statistical mechanics
   a. Free field theories as Gaussian processes
   b. Renormalization group for field theories
7. Monte Carlo simulation of these models
   a. Ising model : slow methods (Glauber) and fast methods (SW)
   b. SAW
   c. ??
8. Polymer or cluster expansions