Math 563 - Take Home Final Solutions

1. Let X and Y be independent random variables. Suppose that X is discrete, i.e., it takes on at most countably many values, and Y has a density, i.e., there is a non-negative measurable function $f(x)$ on the real line with $P(Y \le t) = \int_{-\infty}^{t} f(x)\,dx$. Let $Z = X + Y$.
(a) Show that the distribution function of Z is given by

$$F_Z(z) = \sum_x F_Y(z - x) P(X = x)$$

(b) Show that Z has a density and find it.

Solution:
(a) Let x_n be the values that X takes on. By the countable additivity of P and the independence of X and Y,

$$\begin{aligned} F_Z(z) &= P(Z \le z) = \sum_n P(Z \le z, X = x_n) = \sum_n P(x_n + Y \le z, X = x_n) \\ &= \sum_n P(x_n + Y \le z) P(X = x_n) = \sum_n F_Y(z - x_n) P(X = x_n) \end{aligned}$$

(b) By the above

$$\begin{aligned} F_Z(z) &= \sum_n F_Y(z - x_n) P(X = x_n) = \sum_n \int_{-\infty}^{z - x_n} f(u) du\, P(X = x_n) \\ &= \sum_n \int_{-\infty}^{z} f(u - x_n) du\, P(X = x_n) = \int_{-\infty}^{z} \sum_n f(u - x_n) P(X = x_n)\, du \end{aligned}$$

where the last equality is justified by Tonelli's theorem. Thus Z has density

$$f_Z(u) = \sum_n f(u - x_n) P(X = x_n)$$

2. Let X_n be an independent, identically distributed sequence of positive random variables with $E[(\ln(X_n))^2] < \infty$.
(a) Prove that $(X_1 X_2 \cdots X_n)^{1/n}$ converges almost surely to a constant c.
(b) Prove that for all real x,

$$\lim_{n \to \infty} P(c^{-\sqrt{n}} (X_1 X_2 \cdots X_n)^{1/\sqrt{n}} \le x)$$

exists, and find the limit. (You won't be able to give a completely explicity formula for the limit.)

Solution:
(a) Let $Y_n = \ln(X_n)$. Then Y_n is an i.i.d. sequence with finite variance. By the strong law of large numbers,

$$\frac{1}{n}\sum_{k=1}^{n} Y_k \to E[Y_1], \quad a.s.$$

Since

$$\ln\left[(X_1 X_2 \cdots X_n)^{1/n}\right] = \frac{1}{n}\sum_{k=1}^{n} Y_k$$

this implies $(X_1 X_2 \cdots X_n)^{1/n}$ converges a.s. to $c = \exp(E[\ln(X_1)])$.
(b) Let σ^2 be the variance of Y_1. By the central limit theorem,

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}[Y_k - \ln(c)] \to \sigma Z$$

where Z has a standard normal distribution and the convergence is in distribution. So

$$P(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}[Y_k - \ln(c)] \le x) \to P(\sigma Z \le x)$$

Since

$$\exp\left[\frac{1}{\sqrt{n}}\sum_{k=1}^{n}[Y_k - \ln(c)]\right] = c^{-\sqrt{n}}(X_1 X_2 \cdots X_n)^{1/\sqrt{n}}$$

this shows that $P(c^{-\sqrt{n}}\,(X_1 X_2 \cdots X_n)^{1/\sqrt{n}} \le x)$ converges to $P(\sigma Z \le \ln(x))$ where Z has a standard normal distribution.

3. Let X_n and X be random variables on the same probability space. There are three types of convergence: (1) pointwise convergence almost surely, (2) convergence in probability and (3) convergence in distribution.
(a) For each of the three types of convergence prove or disprove that if X_n converges to X then $E[X_n]$ converges to $E[X_n]$.

(b) For each of the three types of convergence prove or disprove that if X_n converges to X then $E[\exp(-X_n^2)]$ converges to $E[\exp(-X^2)]$.

Solution:
(a) False for all three notions of convergence. Take the probability space to be $[0,1]$ with Lebesgue measure. Let $X_n = n1_{(0,1/n)}$. Then X_n converges to 0 pointwise (and so in probability and distribution), but $E[X_n] = 1$ for all n.
(b) True for all three notions of convergence. Since convergence a.s. implies convergence in distribution and convergence in probability implies convergence in distribution, it suffices to show it is true for convergence in distribution. If X_n converges to X in distribution, then we can find another probability space with random variables X_n' and X' such that X_n' equals X_n in distribution, X' equals X in distribution and $X_n' \to X'$ a.s. Note that for all n, $\exp(-(X_n')^2) \leq 1$. Since X_n equals X_n' in distribution, $E[\exp(-X_n^2)] = E[\exp(-(X_n')^2)]$ by the law of the unconscious statistician. Likewise, $E[\exp(-X^2)] = E[\exp(-(X')^2)]$. So by the dominated convergence theorem (or bounded convergence theorem),

$$\begin{aligned} \lim_{n\to\infty} E[\exp(-X_n^2)] &= \lim_{n\to\infty} E[\exp(-(X_n')^2)] = E[\lim_{n\to\infty} \exp(-(X_n')^2)] \\ &= E[\exp(-(X')^2)] = E[\exp(-X^2)] \end{aligned}$$

4. A random variable X has a geometric distribution if it takes values in the non-negative integers and $P(X = k) = (1-p)^k p, k = 0, 1, 2, \cdots$. Let $\alpha > 0$. Let X_n be an independent sequence of random variables such that X_n has the geometric distribution with $p = \alpha/n$. Let $Y_n = X_n/n$. Prove that Y_n converges in distribution and find the limiting distribution.

Solution: By the continuity theorem we can show Y_n converges in distribution by showing their characteristic functions converges pointwise.

First we compute the characteristic function of a geometric RV:

$$E[\exp(iXt)] = \sum_{k=0}^{\infty} (1-p)^k p e^{ikt} = \frac{p}{1-(1-p)e^{it}}$$

Thus

$$\beta_n(t) = E[\exp(itY_n)] = E[\exp(itX_n/n)] = \frac{\alpha/n}{1-(1-\alpha/n)e^{it/n}}$$

For a fixed t, $e^{it/n} = 1 + it/n + O(1/n^2)$. So

$$\begin{aligned}\beta_n(t) &= \frac{\alpha/n}{1-(1-\alpha/n)(1+it/n+O(1/n^2))} = \frac{\alpha/n}{\alpha/n - it/n + O(1/n^2)} \\ &= \frac{\alpha}{\alpha - it + O(1/n)} \to \frac{\alpha}{\alpha - it}\end{aligned}$$

which is the characteristic function of an exponential random variable with parameter α.

5. Let Y be a discrete random variable which takes on the values $\{y_n\}_{n=1}^\infty$. Prove that

$$E[X|\sigma(Y)] = \sum_{n=1}^\infty 1_{Y=y_n} \frac{E[X 1_{Y=y_n}]}{P(Y=y_n)}$$

Solution: Let

$$Z = \sum_{n=1}^\infty 1_{Y=y_n} \frac{E[X 1_{Y=y_n}]}{P(Y=y_n)}$$

To prove $E[X|\sigma(Y)] = Z$ we must prove two things: (1) Z is measurable with respect to $\sigma(Y)$, and (2) $E[Z1_A] = E[X1_A]$ for all $A \in \sigma(Y)$.

Z is a countable sum of $1_{Y=y_n}$ times constants. So to show it is $\sigma(Y)$ measurable, it suffices to show $1_{Y=y_n}$ is $\sigma(Y)$ measurable. This is trivial. So (1) holds.

To prove (2), first suppose $A = \{Y = y_m\}$. Then using Tonelli's thm

$$\begin{aligned}E[Z1_A] &= E[1_A \sum_{n=1}^\infty 1_{Y=y_n} \frac{E[X 1_{Y=y_n}]}{P(Y=y_n)}] \\ &= \sum_{n=1}^\infty \frac{E[X 1_{Y=y_n}]}{P(Y=y_n)} E[1_A \, 1_{Y=y_n}]\end{aligned}$$

If $m \neq n$, then $E[1_A \, 1_{Y=y_n}]$ is 0. If $m = n$, then $E[1_{Y=y_n}]$ is $P(Y = y_m)$. So the above becomes $E[X1_{Y=y_m}] = E[X1_A]$ as needed.

The elements of $\sigma(Y)$ are countable or finite unions of events of the form $\{Y = y_n\}$, so (2) follows for general $A \in \sigma(Y)$.

6. Let X be a random variable with $E[X^2] < \infty$. Let $\mathcal{G}$ be a σ-subfield of $\mathcal{F}$. The conditional variance of X given $\mathcal{G}$ is the random variable defined by

$Var(X|\mathcal{G}) = E[X^2|\mathcal{G}] - (E[X|\mathcal{G}])^2$ Let X_k be an independent sequence of random variables with $E[X_k^2] < \infty$. Define $S_n = \sum_{k=1}^n X_k$, and let $\mathcal{F}_n$ be the σ-field generated by $X_1, X_2, \cdots, X_n$. Prove that $Var(S_{n+1}|\mathcal{F}_n) = Var(X_{n+1})$ a.s.

Solution: Since $X_k \in \mathcal{F}_n$ for $k \leq n$, we have $S_n \in \mathcal{F}_n$. Also, X_{n+1} is independent of $\mathcal{F}_n$. So using properties (1),(2) and (3) of conditional expectation,

$$E[S_{n+1}|\mathcal{F}_n] = E[S_n + X_{n+1}|\mathcal{F}_n] = E[S_n|\mathcal{F}_n] + E[X_{n+1}|\mathcal{F}_n] = S_n + E[X_{n+1}]$$

Now look at the second moment:

$$\begin{aligned} E[S_{n+1}^2|\mathcal{F}_n] &= E[(S_n + X_{n+1})^2|\mathcal{F}_n] = E[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2|\mathcal{F}_n] \\ &= E[S_n^2|\mathcal{F}_n] + E[2S_nX_{n+1}|\mathcal{F}_n] + E[X_{n+1}^2|\mathcal{F}_n] \\ &= S_n^2 + 2S_nE[X_{n+1}|\mathcal{F}_n] + E[X_{n+1}^2|\mathcal{F}_n] = S_n^2 + 2S_nE[X_{n+1}] + E[X_{n+1}^2] \end{aligned}$$

Thus

$$\begin{aligned} Var(S_{n+1}|\mathcal{F}_n) &= E[S_{n+1}^2|\mathcal{F}_n] - (E[S_{n+1}|\mathcal{F}_n])^2 \\ &= S_n^2 + 2S_nE[X_{n+1}] + E[X_{n+1}^2] - (S_n + E[X_{n+1}])^2 \\ &= E[X_{n+1}^2] - (E[X_{n+1}])^2 = var(X_{n+1}) \end{aligned}$$