Math 563 - Homework 1

1. Prove the following proposition. (It was stated in class.)
Proposition: Let $X : \Omega \to \Omega'$ where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are measureable spaces. Suppose that $\mathcal{E} \subset \mathcal{F}'$ and $\mathcal{E}$ generates $\mathcal{F}'$. Suppose also that $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{E}$. Prove that X is measurable.

2. Let X be a real valued function on Ω and let $\sigma(X)$ be the σ-field generated by the sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. (This is the smallest σ-field with respect to which X is measurable.) Let Y be a real valued function on Ω. Prove that Y is measurable with respect to $\sigma(X)$ if and only if there is a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

3. Let E_n be a sequence of events. We define a new event :

$$\{\omega : \omega \in E_n \ for\ infinitely\ many\ n\}$$

This event is sometimes written E_n i. o., where i. o. stands for "infinitely often."
(a) Show that $E_n \, i.o. = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$
(b) Prove that if $\sum_{n=1}^{\infty} P(E_n) < \infty$, then $P(E_n \, i.o.) = 0$. This is sometimes called the "easy half" of the Borel Cantelli lemma.

4. Let X be a simple random variable on a probability space $(\Omega, \mathcal{F}, P)$. Let $c_1, c_2, \cdots, c_n$ be the values that X takes on. Let $p_j = P(X = c_j)$. Let μ_X be the distribution of X. (Recall this means μ_X is the probability measure on $(\mathbb{R}, \mathcal{B})$ defined by $\mu_X(E) = P(X \in E)$ for Borel subsets E of the reals.) Give an explicit description of μ_X in terms of the c_j and p_j. (This is not a hard problem.)

5. We flip a fair coin infinitely many times. Let X_n be 1 if the nth flip is heads, and 0 if the nth flip is tails. The sample space Ω consists of all sequences of heads and tails. X_n is a real valued function on Ω. In this problem we assume that there is a σ-field $\mathcal{F}$ and a probability measure P such that X_n is a random variable and the probability measure agrees with your intiution. (We will eventually prove such an $\mathcal{F}$ and P exist.) Define

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

Note that $0 \leq X \leq 1$. Find the distribution μ_X of X. Hint: find $P(X \in E)$ when E is an interval of the form $((k-1)/2^n, k/2^n)$ for integers k and n.