Math 563 - Homework 1

1. Define $\mathcal{G} = \{B \in \mathcal{F}' : X^{-1}(B) \in \mathcal{F}\}$. Routine set manipulations show that $\mathcal{G}$ is a σ-field. By hypothesis, $\mathcal{G}$ contains $\mathcal{E}$. So it contains the σ-field generated by $\mathcal{E}$, which by hypothesis is $\mathcal{F}'$. This shows X is measurable.

2. First we note that the collection of sets $X^{-1}(B)$ where B is a Borel set in $\mathbb{R}$ is already a σ-field. So $\sigma(X) = \{X^{-1}(B) :\text{B } is\ Borel\ set\ in \mathbb{R}\}$.

If Y is $\sigma(X)$ measurable, then its positive and negative parts are too. So we might as well assume that Y is non-negative. Let

$$
\begin{aligned}
B_{n,k} &= Y^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n})), \quad k = 1, 2, \cdots, n2^n \\
B_n &= Y^{-1}([n, -\infty])
\end{aligned}
$$

Now let

$$Y_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{B_{n,k}} + n 1_{B_n}$$

as we did in class. So this is an increasing sequence of nonnegative simple functions that converges pointwise to Y. Since Y is $\sigma(X)$ measurable, $B_{n,k}$ and B_n are in $\sigma(X)$. So by the first paragraph, there are Borel sets $C_{n,k}$ and C_n such that $B_{n,k} = X^{-1}(C_{n,k})$ and $B_n = X^{-1}(C_n)$. Define $f_n(x)$ by

$$f_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{C_{n,k}} + n 1_{C_n}$$

Note that this is Borel measurable since $C_{n,k}$ and C_n are Borel sets. However, note that the $C_{n,k}$ need not be disjoint. (You can add or substract points that are not in the range of X to the $C_{n,k}$ and C_n and still have $B_{n,k} = X^{-1}(C_{n,k})$ and $B_n = X^{-1}(C_n)$.) This means that f_n is not necessarily increasing. So we do not know that f_n converges pointwise. So we define $f(x) = \limsup_{n\to\infty} f_n(x)$. Since the f_n are Borel measurable, f is too.

Fix an n. We claim that for all $\omega \in \Omega$, we have $f_n(X(\omega)) = Y_n(\omega)$. ω belongs to exactly one of the events $B_{n,k}$, $k = 1, 2, \cdots, n2^n$ and B_n. Suppose it is in $B_{n,k}$ (The other case is similar.) So $Y_n(\omega) = (k-1)/2^n$.

$$\omega \in B_{n,k} \Rightarrow \omega \in X^{-1}(C_{n,k}) \Rightarrow X(\omega) \in C_{n,k}$$

So $f_n(X(\omega))$ is at least $(k-1)/2^n$. Is it possible that $X(\omega)$ is in another $C_{n,j}$ or in C_n? If $X(\omega) \in C_{n,j}$ then $\omega \in X^{-1}(C_{n,j}) = B_{n,j}$. This contradicts

$\omega \in B_{n,k}$. Thus $f_n(X(\omega)) = (k-1)/2^n$, and this proves the claim that $f_n(X) = Y_n$. To finish the proof, we know that for all ω, $Y_n(\omega)$ converges to $Y(\omega)$. So $f_n(X(\omega))$ converges to $Y(\omega)$. But the lim sup of $f_n(X(\omega))$ is $f(X(\omega))$, so $f(X(\omega)) = Y(\omega)$.

3. (a) Let $\omega \in \cap_{n=1}^\infty \cup_{k=n}^\infty E_k$. Then $\forall n$, there is a $k \geq n$ such that $\omega \in E_k$. Let k_1 be such that $\omega \in E_{k_1}$. Now suppose that we have $k_1 < k_2 < \cdots < k_n$ such that $\omega \in E_{k_j}$ for $j = 1, 2, \cdots, n$. Then we can find $k_{n+1} > k_n$ with $\omega \in E_{k_{n+1}}$. By induction, there exists a strictly increasing sequence k_j such that $\omega \in E_{k_j}$ for $j = 1, 2, \cdots$. Thus $\omega \in E_n$ i. o.

Now suppose $\omega \in E_n$ i. o. Then there is a strictly increasing sequence k_j such that $\omega \in E_{k_j}$ for $j = 1, 2, \cdots$. This implies $\omega \in \cup_{k=n}^\infty E_k$ for all n. And this implies $\omega \in \cap_{n=1}^\infty \cup_{k=n}^\infty E_k$.

(b) For all m, E_n i.o $\subset \cup_{k=m}^\infty E_k$. So

$$P(E_n\, i.o.) \leq P(\cup_{k=m}^\infty E_k) \leq \sum_{k=m}^\infty P(E_k)$$

Since $\sum_{k=1}^\infty P(E_k) < \infty$,

$$\lim_{m\to\infty} \sum_{k=m}^\infty P(E_k) = 0$$

4.

$$\mu(E) = \sum_{j:c_j \in E} p_j$$

5. Following the hint, we first compute $P(X \in ((k-1)/2^n, k/2^n))$. Consider

$$S_n = \sum_{j=1}^n \frac{X_j}{2^j}$$

It takes on the values $l/2^n$ where $l = 0, 1, \cdots, 2^n - 1$. Each possible value corresponds to one possible sequence for the first n flips. So each value of S_n has probability $1/2^n$. Now $X = S_n + Y_n$ where

$$Y_n = \sum_{j=n+1}^\infty \frac{X_j}{2^j}$$

Clearly $0 \leq Y_n \leq 2^{-n}$. So $X \in [(k-1)/2^n, k/2^n)$ if and only if $S_n = (k-1)/2^n$. (This is not quite true. If $X_j = 1$ for all $j > n$, then $Y_n = 2^{-n}$. But this only happens with probability zero. So $P(X \in [(k-1)/2^n, k/2^n)) = 2^{-n}$.

By taking unions, we see that So $P(X \in [j/2^n, k/2^n]) = (k-j)2^{-n}$ for $j < k$. We refer to intervals of the form $[j/2^n, k/2^n]$ as "dyadic intervals." Given any interval (a, b), we can find an increasing sequence of dyadic intervals I_m which converges to (a, b) in the sense that $\cup_m I_m = (a, b)$. By continuity of probability this implies $P(a, b) = b - a$. This holds for all $0 \leq a < b \leq 1$, so μ_X is Lebesgue measure on $[0, 1]$.