

## Math 563 - Homework 2

1. We flip a fair coin infinitely many times. The sample space  $\Omega$  consists of all sequences of 0's and 1's, 0 representing tails, 1 heads. We denote the elements of  $\Omega$  by  $(\omega_1, \omega_2, \omega_3, \dots)$ . We define a  $\sigma$ -field  $\mathcal{F}$  as follows. Let  $n$  be a positive integer, and for  $i = 1, 2, \dots, n$  let  $\epsilon_i$  be 0 or 1. Then define

$$E_{\epsilon_1, \dots, \epsilon_n}^n = \{(\omega_1, \omega_2, \omega_3, \dots) : \omega_i = \epsilon_i, \quad i = 1, 2, \dots, n\}$$

Then  $\mathcal{F}$  is the  $\sigma$ -field generated by all the  $E_{\epsilon_1, \dots, \epsilon_n}^n$ . We assume that there is a probability measure  $P$  on  $(\Omega, \mathcal{F})$  such that  $P(E_{\epsilon_1, \dots, \epsilon_n}^n) = 2^{-n}$ . Define a function on  $\Omega$  by

$$X(\omega_1, \omega_2, \omega_3, \dots) = \inf_i \{i : \omega_i = 1\}$$

In words,  $X$  is the number of flips it takes to get heads for the first time. Show that  $X$  is a random variable and find its distribution function.

2. A sequence of real-valued RV's  $X_n$  is said to converge to the real-valued RV  $X$  *in probability* if for every  $\epsilon > 0$  we have

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$$

Show that if  $X_n$  converges to  $X$  almost surely, then  $X_n$  converges to  $X$  in probability. (The converse is not true. ) Hint: use continuity of  $P$ .

3. One of the hypotheses of the monotone convergence theorem is that all the random variables  $X_n$  are non-negative. Prove that the conclusion of the theorem is true without this hypothesis if we assume that  $E[X_1]$  exists and is not  $-\infty$ .

4. Let  $f(x)$  be a non-negative real valued, Borel measurable function on the real line with  $\int_{-\infty}^{\infty} f(x) dx = 1$ . (The integral is with respect to Lebesgue measure.) For Borel subsets  $B$  of the real line, define

$$P(B) = \int_{-\infty}^{\infty} 1_B(x) f(x) dx$$

Use the Monotone convergence theorem to show that  $P$  is a probability measure. (This is straightforward.)

5. Let  $X$  be a random variable whose distribution function is the Cantor function. Calculate  $E[X]$  and  $E[X^2]$ . You should justify your calculation.

We know that for any random variable and a real-valued measurable function  $f$  on the real line,

$$E[f(X)] = \int_{\mathbb{R}} f(x) d\mu_X$$

where  $\mu_X$  is the distribution of  $X$ . The point of the next two problems is to get more explicit formulae that are amenable to computation in the cases that  $X$  is discrete or has a density.

6. Let  $X$  be a discrete real-valued random variable. Let  $x_1, x_2, \dots$  be its values and let  $p_n = P(X = x_n)$ . Let  $g$  be any real valued function on the real line. Suppose that

$$\sum_n |g(x_n)| p_n < \infty$$

Prove that  $g(X)$  is a random variable and

$$E[g(X)] = \sum_n g(x_n) p_n$$

Note that I did not say that  $g$  was measurable.

7. Let  $X$  be a random variable which has a density  $f(x)$ . Recall this means its distribution function  $F(x)$  satisfies  $F(x) = \int_{-\infty}^x f(u) du$ . Let  $g(x)$  be a real-valued measurable function on the real line such that

$$\int_{-\infty}^{\infty} |g(x)| f(x) dx < \infty$$

Prove that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

where the integrals with respect to  $dx$  are integration with respect to Lebesgue measure.

*The last three problems are **optional**. They are standard problems in a course like 564. I have included them for background.*

8. (564 discrete RV's) Let  $X$  be a discrete RV,  $x_1, x_2, \dots$  its values and  $p_n = P(X = x_n)$ . The function  $f(x)$  which is  $p_n$  at  $x_n$  and is 0 at points not

equal to one of the  $x_n$  is often called the “probability mass function” in 564 level prob. courses.

Suppose we a coin with probability  $p$  of heads. ( $p$  is not necessarily  $1/2$ .) The following discrete RV’s are of interest:

**Binomial:** Fix a positive integer  $n$ . Flip the coin  $n$  times and let  $X$  be the number of heads. So  $X$  can be  $0, 1, 2, \dots, n$ .

**Geometric:** Flip the coin until you get heads for the first time. Let  $X$  be the number of tails. (Warning: depending on the author the definition of  $X$  is sometimes taken to be the total number of flips, including the final one that gave heads.) So the possible values of  $X$  are the nonnegative integers.

**Negative binomial:** Fix an integer  $r$ . Flip the coin until we get heads for the  $r$ th time. Let  $X$  be the total number of tails. So the possible values of  $X$  are the nonnegative integers.

For each of these find the probability mass function and the expected value of  $X$ .

9. (564 continuous RV’s) In a 564 level course, “continuous RV” usually means that the RV has a density  $f(x)$ . (Having a density implies the distribution function is continuous, but the converse is very false.) Here are three common “continuous” RV’s.

**Uniform:** We say  $X$  is uniform on  $[a, b]$  if  $f(x) = \frac{1}{b-a}$  for  $a \leq x \leq b$  and  $f(x) = 0$  otherwise.

**Exponential:** We say  $X$  has an exponential distribution if  $f(x) = \lambda e^{-\lambda x}$  for  $x \geq 0$  and  $f(x) = 0$  for  $x < 0$ . Here  $\lambda$  is a positive parameter.

**Normal:** We say  $X$  has a normal distribution if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

Here  $\mu$  is a real parameter and  $\sigma^2$  is a positive parameter.

Find the expected value of each of these random variables.

10. Let  $X$  be a random variable with a normal distribution. Find the density function of  $X^2$ . Hint: first find an expression for the distribution function of  $X^2$ .