Math 563 - Homework 2 - Solutions

2. A sequence of real-valued RV's X_n is said to converge to the real-valued RV X *in probability* if for every $\epsilon > 0$ we have

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$$

Show that if X_n converges to X almost surely, then X_n converges to X in probability. (The converse is not true.) Hint: use continuity of P.

Solution:

Let A be the set of ω where $X_n(\omega)$ does not converge to $X(\omega)$. So $P(A) = 0$. Now let $\epsilon > 0$. Suppose that ω is such that for every N we can find $n \geq N$ such that $|X_n(\omega) - X(\omega)| \geq \epsilon$. Then $X_n(\omega)$ does not converge to $X(\omega)$. Thus

$$\cap_{N=1}^{\infty} \cup_{n=N}^{\infty} \{|X_n(\omega) - X(\omega)| \geq \epsilon\} \subset A$$

Since $\cup_{n=N}^{\infty}\{|X_n(\omega) - X(\omega)| \geq \epsilon\}$ is a decreasing sequence in N, by the continuity of P we have

$$\begin{aligned} \lim_{N\to\infty} \quad & P(\cup_{n=N}^{\infty}\{|X_n(\omega) - X(\omega)| \geq \epsilon\}) = P(\cap_{N=1}^{\infty} \cup_{n=N}^{\infty} \{|X_n(\omega) - X(\omega)| \geq \epsilon\}) \\ \leq \quad & P(A) = 0 \end{aligned}$$

Since

$$P(\{|X_N(\omega) - X(\omega)| \geq \epsilon\}) \leq P(\cup_{n=N}^{\infty}\{|X_n(\omega) - X(\omega)| \geq \epsilon\})$$

we have shown that

$$\lim_{N\to\infty} P(\{|X_N(\omega) - X(\omega)| \geq \epsilon\}) = 0$$

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3. One of the hypotheses of the monotone convergence theorem is that all the random variables X_n are non-negative. Prove that the conclusion of the theorem is true without this hypothesis if we assume that $E[X_1]$ exists and is not $-\infty$.

Solution:

First suppose that $E[X_1] = +\infty$. Then $X_1 \leq X_n$ implies all $E[X_n] = +\infty$. So this case is trivial .

Now suppose $E[X_1] < \infty$. We also know that $E[X_1]$ is not $-\infty$. Thus X_1 is finite a.s. For ω such that $X_1(\omega)$ is finite, we define

$$Y_n(\omega) = X_n(\omega) - X_1(\omega)$$

For the other ω we define $Y_n(\omega) = 0$. Now Y_n is increasing and nonnegative. It converge to $X - X_1$ a.s. So we can apply the usual monontone convergence theorem to this sequence. So

$$\lim_{n\to\infty} E[Y_n] = E[X - X_1]$$

Since $E[X_1]$ is finite, $E[X - X_1] = E[X] - E[X_1]$ and $E[Y_n] = E[X_n - X_1] = E[X_n] - E[X_1]$. The result follows.

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4. Let $f(x)$ be a non-negative real valued, Borel measurable function on the real line with $\int_{-\infty}^{\infty} f(x)\, dx = 1$. (The integral is with respect to Lebesgue measure.) For Borel subsets B of the real line, define

$$P(B) = \int_{-\infty}^{\infty} 1_B(x)\, f(x)\, dx$$

Use the Monotone convergence theorem to show that P is a probability measure. (This is straightforward.)

Solution:

We have

$$P(\Omega) = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Next we check countable additivity. Let B_n be a disjoint sequence of events. Define

$$X = 1_{\cup_{n=1}^{\infty} B_n}$$

and

$$X_n = 1_{\cup_{k=1}^{n} B_k} = \sum_{k=1}^{n} 1_{B_k}$$

Then X_n is an increasing sequence that converges to X. By the monotone convergence theorem,

$$P(\cup_{n=1}^{\infty} B_n) = E[X] = \lim_{n\to\infty} E[X_n] = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k) = \sum_{k=1}^{\infty} P(B_k)$$

Thus P is a probability measure.

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5. Let X be a random variable whose distribution function is the Cantor function. Calculate $E[X]$ and $E[X^2]$. You should justify your calculation.

Solution:

Let μ_X be the distribution of X. By the law of the unconscious statistician, we can compute $\int x d\mu_X$ and $\int x^2 d\mu_X$. Note that $\mu(-\infty, 0] = \mu[1, \infty) = 0$. So we can define simply functions that approximate x by

$$f_n(x) = \sum_{k=1}^{3^n} \frac{k-1}{3^n} \, 1_{[\frac{k-1}{3^n}, \frac{k}{3^n})}(x)$$

for $x \in [0, 1]$ and $f_n(x) = 0$ for $x \notin [0, 1]$. The the f_n are increasing and converge to x, μ_X a.s. So by monotone convergence theorem,

$$E[X] = \int x \, d\mu_X = \lim_{n\to\infty} \int f_n(x) \, d\mu_X$$

We have

$$\int f_n(x) \, d\mu_X = \sum_{k=1}^{3^n} \frac{k-1}{3^n} \left[F(\frac{k}{3^n}) - F(\frac{k-1}{3^n}) \right]$$

The value of the $F(\frac{k}{3^n})$ is determined at the nth stage of the definition of the Cantor function. And $F(\frac{k}{3^n}) - F(\frac{k-1}{3^n})$ is either 0 or 2^{-n}. To determine which we can use ternary (base 3) expansions: Let

$$\frac{k-1}{3^n} = 0.a_1 a_2 a_3 \cdots a_n 000 \cdots$$

where each a_i is $0, 1$ or 2 for $i = 1, 2, \cdots, n$. So for $x \in [\frac{k-1}{3^n}, \frac{k}{3^n})$

$$x = 0.a_1 a_2 a_3 \cdots a_n a_{n+1} a_{n+1} \cdots$$

Note that $a_i = 1$ means that at the ith stage x is in a "middle third interval" on which F is constant. So $F(\frac{k}{3^n}) - F(\frac{k-1}{3^n})$ is 2^{-n} if and only if $a_1, a_2, \cdots, a_n$ take on only the values 0 or 2. Note that (1) says

$$\frac{k-1}{3^n} = \sum_{k=1}^{n} \frac{a_k}{3^k}$$

Thus (1) is equal to

$$\begin{aligned}
\sum_{a_1,a_2,\cdots,a_n=0,2} \sum_{k=1}^{n} \frac{a_k}{3^k} 2^{-n} &= 2^{-n} \sum_{k=1}^{n} \sum_{a_1,a_2,\cdots,a_n=0,2} \frac{a_k}{3^k} \\
&= 2^{-n} \sum_{k=1}^{n} 2^{n-1} \sum_{a_k=0,2} \frac{a_k}{3^k} \\
&= \frac{1}{2} \sum_{k=1}^{n} \frac{2}{3^k} \\
&= \sum_{k=1}^{n} \frac{1}{3^k}
\end{aligned}$$

Letting $n \to \infty$ and summing the geometric series we get 1/2.

To compute $E[X^2]$ we can use the same sequence of simple functions: f_n^2 increases to x^2 so by monotone convergence theorem

$$\begin{aligned}
E[X^2] &= \int x^2 \, d\mu_X = \lim_{n\to\infty} \int f_n^2(x) \, d\mu_X \\
&= \sum_{a_1,a_2,\cdots,a_n=0,2} \left(\sum_{k=1}^{n} \frac{a_k}{3^k} \right)^2 2^{-n} \\
&= \sum_{a_1,a_2,\cdots,a_n=0,2} \sum_{k,l=1}^{n} \frac{a_k a_l}{3^{k+l}} 2^{-n} \\
&= \sum_{k,l=1}^{n} \sum_{a_1,a_2,\cdots,a_n=0,2} \frac{a_k a_l}{3^{k+l}} 2^{-n} \\
&= \sum_{1\le k,l\le n : k\ne l} \sum_{a_1,a_2,\cdots,a_n=0,2} \frac{a_k a_l}{3^{k+l}} 2^{-n} + \sum_{1\le k\le n} \sum_{a_1,a_2,\cdots,a_n=0,2} \frac{a_k^2}{3^{2k}} 2^{-n}
\end{aligned}$$

$$= \sum_{1\le k,l\le n: k\ne l} \frac{1}{3^{k+l}} + 2\sum_{1\le k\le n} \frac{1}{3^{2k}}$$
$$= \sum_{1\le k,l\le n} \frac{1}{3^{k+l}} + \sum_{1\le k\le n} \frac{1}{3^{2k}} \to 3/8$$

as $n \to \infty$

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6. Let X be a discrete real-valued random variable. Let $x_1, x_2, \cdots$ be its values and let $p_n = P(X = x_n)$. Let g be any real valued function on the real line. Suppose that

$$\sum_n |g(x_n)|\, p_n < \infty$$

Prove that $g(X)$ is a random variable and

$$E[g(X)] = \sum_n g(x_n) p_n$$

Note that I did not say that g was measurable.

Solution:

Since X takes on at most a countable number of values, $g(X)$ takes on at most a countable number of values, call them $y_1, y_2, \cdots$. Now let B be a Borel set in $\mathbb{R}$. Then $(g(X))^{-1}(B) = (g(X))^{-1}(B')$ where B' is $B \cap \{y_1, y_2, \cdots\}$. Now

$$(g(X))^{-1}(B') = \cup_{y\in B'} (g(X))^{-1}(\{y\})$$

and since B' is countable it suffices to show $(g(X))^{-1}(\{y\})$ is an event. It is equal to $\cup_{x:g(x)=y} X^{-1}(\{x\})$. We can restrict the union to the x in the range of X and so this union is at most coutable. Each $X^{-1}(\{x\})$ is measurable, so this completes the proof that $g(X)$ is a random variable.

Let $x_1, x_2, \cdots$ be the values of X. Let $E_i = \{X = x_i\}$. So E_i is a partition of Ω. Suppose g is nonnegative. Define

$$Y_n = \sum_{i=1}^n g(x_i)\, 1_{E_i}$$

Then Y_n is a simple function, and the Y_n increase to $g(X)$. (Note that for this sequence to be increasing it is essential that g be nonnegative.) By monotone convergence

$$E[g(X)] = \lim_{n\to\infty} E[Y_n] = \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\, P(E_i) = \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\, p_i = \sum_{i=1}^{\infty} g(x_i)\, p_i$$

For general g we write it as $g = g^+ - g^-$ where g^+ and g^- are nonnegative. Then $g(X) = g^+(X) - g^-(X)$ and we can apply the above result to $g^+(X)$ and $g^-(X)$.

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7. Let X be a random variable which has a density $f(x)$. Recall this means its distribution function $F(x)$ satisfies $F(x) = \int_{-\infty}^{x} f(u)\, du$. Let $g(x)$ be a real-valued measurable function on the real line such that

$$\int_{-\infty}^{\infty} |g(x)|\, f(x)\, dx < \infty$$

Prove that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

where the integrals with respect to dx are integration with respect to Lebesgue measure.

Solution:

By the law of the unconscious statistician, $E[g(X)] = \int g(x)\, d\mu_X$. So it suffice to show $\int g(x)\, d\mu_X = \int g(x)\, f(x)\, dx$.

By problem 4, we can define a probability measure μ on the real line by

$$\mu(B) = \int 1_B(x)\, f(x)\, dx$$

We claim that $\mu = \mu_X$, where μ_X is the distribution for the random variable X. By the uniqueness theorem it suffice to show that the two measures agree on a collection of events which is closed under intersections and which generates the σ-field of Borel sets. The collection of intervals of $(a, b]$ is such a collection of events. We have

$$\mu_X((a,b]) = F(b) - F(a) = \int_{-\infty}^{b} f(x)\, dx - \int_{-\infty}^{a} f(x)\, dx = \int_{a}^{b} f(x)\, dx = \mu((a,b])$$

Thus $\mu = \mu_X$.

Now let $g(x)$ be a simple function from $\mathbb{R}$ to $\mathbb{R}$:

$$g(x) = \sum_{i=1}^{n} c_i \, 1_{B_i}(x)$$

Then

$$\int g(x)\, d\mu_X = \sum_{i=1}^{n} c_i \, \mu_X(B_i) = \sum_{i=1}^{n} c_i \, \mu(B_i) = \sum_{i=1}^{n} c_i \int 1_{B_i}(x)\, f(x)\, dx = \int g(x)\, f(x)\, dx$$

Now let $g(x)$ be a nonnegative Borel measure function from $\mathbb{R}$ to $\mathbb{R}$, and let $g_n(x)$ be nonnegative simple functions which increase to $g(x)$. Then by the monotone convergence theorem,

$$\int g(x)\, d\mu_X = \lim_{n\to\infty} \int g_n(x)\, d\mu_X = \lim_{n\to\infty} \int g_n(x)\, d\mu = \int g(x)\, f(x)\, dx$$

We complete the proof in the usual way by writing $g = g^+ - g^-$.