Math 563 - Homework 3

1. Let $\Omega = \{a, b, c, d\}$. Let $\mathcal{F}$ be the σ-field containing all subsets of Ω. Let $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$. Show that $\sigma(\mathcal{E}) = \mathcal{F}$ but there exist two different probability measures on $(\Omega, \mathcal{F})$ which agree on $\mathcal{E}$.

2. (text, problem 14 on p. 64) Let X be a real valued random variable. Let ϕ be an increasing real valued function on the real line. (Assume it is Borel measurable.) Suppose that X and $\phi(X)$ both have finite mean and variances. Prove that $Cov(X, \phi(X)) \geq 0$.

3. (text, problem 21, p. 98) Let $\Omega = (0, 1] \times (0, 1]$. Consider the rectangles of the form $(a, b] \times (c, d]$ where $0 \leq a < b \leq 1$ and $0 \leq c < d \leq 1$. Let $\mathcal{E}$ be the collection of sets which are finite unions of such rectangles. For $A \in \mathcal{E}$, define $R(A)$ to be the area of A. Show that $\mathcal{E}$ is a field and R is finitely additive on $\mathcal{E}$. Then show that R is countably additive by using "proposition 9." The resulting extension of R to the Borel sets is two dimensional Lebesgue measure on the unit square. Hint: to prove countable additivity, use proposition 9 and the strategy I used in class when we showed that given a distribution function $F(x)$, there is a probability measure μ with that distribution.

4. (text, problem 2, p. 86) Show that if a Sierpinski class of subsets of Ω is closed under pairwise intersections and contains Ω, then it is a σ-field.

5. Suppose X_n are identically distributed, uncorrelated, and have finite second moment, i.e., $EX_n^2 < \infty$. We proved a weak law of large numbers which says that for any $\epsilon > 0$, the probability

$$P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$$

converges to zero as $n \to \infty$, but we didn't say anything about how fast it converges.

(a) Use the proof from class of the weak law for the case of finite second moment to show it converges to zero at least as fast as $1/n^p$ for some power p. (I.e., show that the probability is $\leq c/n^p$ for some constant c, which can depend on ϵ.) You should find the biggest p you can.

(b) Now suppose that you also know that $EX_n^4 < \infty$ and that $E(X_i X_j X_k X_l) = E(X_i)E(X_j X_k X_l)$ when i is not equal to any of j, k, l, and that $E(X_i X_j X_k X_l) = E(X_i X_j)E(X_k X_l)$ when i is not equal to k or l and j is not equal to k or

l. Prove that $P(|\frac{1}{n}\sum_{k=1}^n X_k - \mu| > \epsilon)$ converges to zero faster than your bound from (a), i.e., with a bigger p. (We have not yet defined independence for a sequence of RV's. When we do, we will see that if the sequence is independent then these equalities involving expectations hold.)

6. *Recurrence of the one-dimensional random walk:* We flip a fair coin infinitely many times. Let $\Omega = \{0,1\}^{\mathbb{N}}$. (We use 0 to indicate tails, 1 for heads.) Let $\mathcal{F}$ and P be the σ-field and probability measure we constructed in class. Define a sequence of random variables by $X_n((\omega_1, \omega_2, \cdots)) = 2\omega_n - 1$. So $X_n = 1$ if nth flip is heads, $= -1$ if nth flip is tails. Let $S_n = \sum_{k=1}^n X_k$. We can think of S_n as the position at time n of a one-dimensional random walk which starts at 0. At each time step you take a step to the right or left, each with probabililty 1/2. The goal of this problem is to use the Borel-Cantelli lemma to prove that $S_n = 0$ i.o. In words, the random walk will return to 0 infinitely many times with probability one.
(a) Define

$$\begin{aligned} A_k &= \{S_k > 0\}, \quad B_k = \{S_k < 0\} \\ A &= A_k \ i.o., \quad B = B_k \ i.o. \end{aligned}$$

Show that $A \cap B \subset \{S_n = 0 \ i.o.\}$. Show that if $P(A) = 1$ and $P(B) = 1$, then $P(S_n = 0 \ i.o.) = 1$.

(b) Chose an increasing sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that $\sqrt{n_k - n_{k-1}} \geq n_{k-1}$. Note that this implies $n_k - n_{k-1}$ goes to infinity. Define events C_k and D_k by

$$\begin{aligned} C_k &= \{S_{n_k} - S_{n_{k-1}} \geq \sqrt{n_k - n_{k-1}}\}, \\ D_k &= \{S_{n_k} - S_{n_{k-1}} \leq -\sqrt{n_k - n_{k-1}}\}, \end{aligned}$$

For this problem you may assume $\lim_{k\to\infty} P(C_k)$ and $\lim_{k\to\infty} P(D_k)$ both exist and are not zero. (This will follow from the central limit theorem. Note that the variance of $S_{n_k} - S_{n_{k-1}}$ is $(n_k - n_{k-1})/4$.) Prove that $P(C_k \ i.o.) = 1$ and $P(D_k \ i.o.) = 1$.

(c) Use the result of (b) to prove $P(A) = 1$ and $P(B) = 1$.

7. **Optional** Compute the variance of all the discrete and "continuous" RV's in problems 8 and 9 of the last homework. Hint: For the discrete RV's there is a trick. Consider the binomial. Let $X = X_1 + X_2 + \cdots X_n$ where X_i is 0 or 1 corresponding to the ith flip. It is easy to compute things like $E[X_i]$ and $E[X_i X_j]$. Use this to compute $E[X]$ and $E[X^2]$.