Math 563 - Homework 3 - Selected solutions

2. (text, problem 14 on p. 64) Let X be a real valued random variable. Let ϕ be an increasing real valued function on the real line. (Assume it is Borel measurable.) Suppose that X and $\phi(X)$ both have finite mean and variances. Prove that $Cov(X, \phi(X)) \geq 0$.

Solution: Since ϕ is increasing, for all real numbers x, y we have

$$(x - y)(\phi(x) - \phi(y) \geq 0$$

Take $x = X(\omega)$ and $y = E[X]$, so

$$(X(\omega) - E[X])(\phi(X(\omega)) - \phi(E[X]) \geq 0$$

i.e.,

$$X(\omega)\phi(X(\omega)) - X(\omega)\phi(E[X]) - E[X]\phi(X(\omega)) + E[X]\phi(E[X])$$

Now take the expected value of this inequality, keeping in mind that $E[X]$ and $\phi(E[X])$ are constants.

$$E[X\phi(X)] - E[X]\phi(E[X]) - E[X]E[\phi(X)] + E[X]\phi(E[X]) \geq 0$$

Simplifying this becomes

$$E[X\phi(X)] - E[X]E[\phi(X)] \geq 0$$

This completes the proof since the above is $Cov(X, \phi(X))$.

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5. Suppose X_n are identically distributed, uncorrelated, and have finite second moment, i.e., $EX_n^2 < \infty$. We proved a weak law of large numbers which says that for any $\epsilon > 0$, the probability

$$P(|\frac{1}{n}\sum_{k=1}^{n} X_k - \mu| > \epsilon)$$

converges to zero as $n \to \infty$, but we didn't say anything about how fast it converges.

(a) Use the proof from class of the weak law for the case of finite second moment to show it converges to zero at least as fast as $1/n^p$ for some power

p. (I.e., show that the probability is $\leq c/n^p$ for some constant c, which can depend on ϵ.) You should find the biggest p you can.

Solution: If you look at the proof from class (or the book) using Chebyshev's inequality you see that we proved that it converges like $1/n$, i.e., $p = 1$.

(b) Now suppose that you also know that $EX_n^4 < \infty$ and that $E(X_i X_j X_k X_l) = E(X_i)E(X_j X_k X_l)$ when i is not equal to any of j, k, l, and that $E(X_i X_j X_k X_l) = E(X_i X_j)E(X_k X_l)$ when i is not equal to k or l and j is not equal to k or l. Prove that $P(|\frac{1}{n}\sum_{k=1}^n X_k - \mu| > \epsilon)$ converges to zero faster than your bound from (a), i.e., with a bigger p. (We have not yet defined independence for a sequence of RV's. When we do, we will see that if the sequence is independent then these equalities involving expectations hold.)

Solution: Let $Y_n = X_n - \mu$, where μ is the common mean of the X_n. Since the X_n are uncorrelated, we have $E(X_i X_j) = 0$ if $i \neq j$. Using the conditions on $E(X_i X_j X_k X_l)$, we find that $E(Y_i Y_j Y_k Y_l)$ vanishes unless $i = j = k = l$ or two of i, j, k, l are equal one value and the other two equal another value. So

$$\begin{aligned} E[(\sum_{i=1}^n Y_i)^4] &= E[\sum_{i,j,k,l=1}^n Y_i Y_j Y_k Y_l] = \sum_{i=1}^n E[Y_i^4] + \sum_{i,j=1:i\neq j}^n E[Y_i^2 Y_j^2] \\ &= nE[Y_1^4] + \frac{n(n-1)}{2} E[Y_1^2]^2 \end{aligned}$$

Now we use a Chebyshev type inequality. Let $S_n = \sum_{i-1}^n Y_i$.

$$E[S_n^4] \geq E[S_n^4 1_{S_n \geq n\epsilon}] \geq P(S_n \geq n\epsilon) n^4 \epsilon^4$$

Putting this together with the above calculation,

$$P(S_n \geq n\epsilon) \leq \frac{nE[Y_1^4] + \frac{n(n-1)}{2} E[Y_1^2]^2}{n^4 \epsilon^4}$$

Since $P(S_n \geq n\epsilon) = P(\frac{1}{n} S_n \geq \epsilon)$, we see that it goes to zero as $1/n^2$. So $p = 2$.

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6. *Recurrence of the one-dimensional random walk:* We flip a fair coin infinitely many times. Let $\Omega = \{0, 1\}^{\mathbb{N}}$. (We use 0 to indicate tails, 1 for heads.) Let $\mathcal{F}$ and P be the σ-field and probability measure we constructed in class. Define a sequence of random variables by $X_n((\omega_1, \omega_2, \cdots)) = 2\omega_n - 1$. So $X_n = 1$ if nth flip is heads, $= -1$ if nth flip is tails. Let $S_n = \sum_{k=1}^n X_k$.

We can think of S_n as the position at time n of a one-dimensional random walk which starts at 0. At each time step you take a step to the right or left, each with probabililty 1/2. The goal of this problem is to use the Borel-Cantelli lemma to prove that $S_n = 0$ i.o. In words, the random walk will return to 0 infinitely many times with probability one.
(a) Define

$$\begin{aligned} A_k &= \{S_k > 0\}, \quad B_k = \{S_k < 0\} \\ A &= A_k \ i.o., \quad B = B_k \ i.o. \end{aligned}$$

Show that $A \cap B \subset \{S_n = 0 \ i.o.\}$. Show that if $P(A) = 1$ and $P(B) = 1$, then $P(S_n = 0 \ i.o.) = 1$.

(b) Chose an increasing sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that $\sqrt{n_k - n_{k-1}} \geq n_{k-1}$. Note that this implies $n_k - n_{k-1}$ goes to infinity. Define events C_k and D_k by

$$\begin{aligned} C_k &= \{S_{n_k} - S_{n_{k-1}} \geq \sqrt{n_k - n_{k-1}}\}, \\ D_k &= \{S_{n_k} - S_{n_{k-1}} \leq -\sqrt{n_k - n_{k-1}}\}, \end{aligned}$$

For this problem you may assume $\lim_{k\to\infty} P(C_k)$ and $\lim_{k\to\infty} P(D_k)$ both exist and are not zero. (This will follow from the central limit theorem. Note that the variance of $S_{n_k} - S_{n_{k-1}}$ is $(n_k - n_{k-1})/4$.) Prove that $P(C_k \ i.o.) = 1$ and $P(D_k \ i.o.) = 1$.

(c) Use the result of (b) to prove $P(A) = 1$ and $P(B) = 1$.

Solutions: (a) Let $\omega \in A \cap B$. So ω belongs to infinitely many A_k and infinitely many B_k. By induction we can construct sequences $n_1 < n_2 < n_3 < \cdots$ and $m_1 < m_2 < m_3 < \cdots$ with $n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \cdots$ and $\omega \in A_{n_k}$ and $\omega \in B_{n_k}$. Because $X_i = \pm 1$, if $S_{n_k} > 0$ and $S_{m_k} < 0$ then there must exist l between n_k and m_k such that $S_l = 0$. So $S_n = 0$ i.o.

The intersection of any two sets with probability 1 has probability 1. In fact, the intersection of a countable number of sets with probability 1 has probability 1. This follows by looking at complements and using the fact that the countable union of sets with probability 0 has probability 0.

(b) For $k \neq l$, $S_{n_k} - S_{n_{k-1}}$ and $S_{n_j} - S_{n_{j-1}}$ depend on disjoint sets of X_i's. So they are independent. So C_k and C_j are independent. By the Borel-Cantelli lemma, $P(C_k \, i.o.) = 1$. Similarly, $P(D_k \, i.o.) = 1$.

WARNING: You can show that the expected value of the product of $S_{n_k} - S_{n_{k-1}}$ and $S_{n_j} - S_{n_{j-1}}$ is zero, as are their individual expected values. So

these two RV's are uncorrelated. This does not imply that the events C_j and C_k are independent.
(c) Choose the sequence n_k so that $\sqrt{n_k - n_{k-1}} \geq n_{k-1}$. The key point is that since $X_i = \pm 1$, we always have $|S_n| \leq n$. So if

$$S_{n_k} - S_{n_{k-1}} \geq \sqrt{n_k - n_{k-1}} > n_{k-1}$$

then we have $S_{n_k} > n_{k-1} + S_{n_{k-1}} \geq 0$. So $C_k \subset A_{n_k}$. So if C_k happens i.o., then so does A_n.