Math 563 - Homework 4

1. Let $X_1, X_2, \cdots, X_n$ be real-valued random variables on a common probability space. Prove they are independent if and only if

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \cdots, x_n \in (-\infty, \infty]$.

2. Let X be a RV. Let $p \geq 1$. Prove that

$$E|X|^p = \int_0^\infty p x^{p-1} P(|X| \geq x) dx$$

Hint: Write $P(|X| \geq x)$ as the expected value of an indicator function. Then you should start seeing double (integrals that is).

3. Let X and Y be random variables on a common probability space. Let $\mu_{X,Y}$ denote their joint distribution. So this is a probability measure on $\mathbb{R}^2$. Let μ_X and μ_Y be the distributions of X and Y, respectively. (So these are both probability measure on $\mathbb{R}$.) Prove that X and Y are independent if and only if $\mu_{X,Y} = \mu_X \times \mu_Y$.

4. Let X_n be a sequence of random variables which converges to the random variable X a.s. Suppose there is a $p > 1$ such that

$$\sup_n E[|X_n|^p] < \infty$$

Prove that $E[|X_n - X|]$ converges to 0.

5. A nonnegative random variable has an exponential distribution if it has density $f(x) = \lambda \exp(-\lambda x)$ for some positive parameter λ. ($f(x) = 0$ if $x < 0$.) Let X and Y be independent random variables, each of which has an exponential distribution with means μ_1 and μ_2, respectively. Let $X = \min\{X, Y\}$. Find the density of X. Hint: first find the distribution function of X.

6. We flip a fair coin infinitely many times. The sample space Ω consists of all sequences of 0's and 1's, 0 representing tails, 1 heads. We denote the elements of Ω by $(\omega_1, \omega_2, \omega_3, \cdots)$. We define a σ-field $\mathcal{F}$ as we did in class.

Let n be a positive integer, and for $i = 1, 2, \cdots, n$ let ϵ_i be 0 or 1. Then define

$$E^n_{\epsilon_1, \cdots, \epsilon_n} = \{(\omega_1, \omega_2, \omega_3, \cdots) : \omega_i = \epsilon_i, \quad i = 1, 2, \cdots, n\}$$

Then $\mathcal{F}$ is the σ-field generated by all the $E^n_{\epsilon_1, \cdots, \epsilon_n}$. We proved in class that there is a probability measure P on $(\Omega, \mathcal{F})$ such that $P(E^n_{\epsilon_1, \cdots, \epsilon_n}) = 2^{-n}$. Let X_n be the random variable

$$X_n(\omega_1, \omega_2, \cdots) = \omega_n$$

(So X_n is 0 or 1 depending on the nth flip.) Show that $\{X_n\}_{n=1}^{\infty}$ is independent. (This is intuitively obvious. The point of the problem is to show it using the definition of independence of RV's.)