Math 563 - Homework 4

1. Let $X_1, X_2, \cdots, X_n$ be real-valued random variables on a common probability space. Prove they are independent if and only if

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \cdots, x_n \in (-\infty, \infty]$.

SOLUTION: Recall that the definition of $X_1, X_2, \cdots, X_n$ being independent is that
$\{\sigma(X_1), \sigma(X_2), \cdots, \sigma(X_n)\}$ is independent. And $\sigma(X_i)$ is the collection of sets of the form $X_i^{-1}(B)$ where B is a Borel set in $\mathbb{R}$. In particular $\sigma(X_i)$ contains the event $X_i \leq x_i$. So $X_1, X_2, \cdots, X_n$ being independent immediately implies that

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \cdots, x_n \in (-\infty, \infty]$.

Now suppose that we have

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \cdots, x_n \in (-\infty, \infty]$. We must show

$$P(X_1 \in B_1, X_2 \in B_2, \cdots, X_n \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i)$$

for all Borel sets $B_1, B_2, \cdots, B_n$.

Define ASSERTION(k) to be the statement that for all $x_{k+1}, x_{k+2}, \cdots, x_n \in (-\infty, \infty]$ and all Borel sets $B_1, B_2, \cdots B_k$ we have

$$P(X_1 \in B_1, X_2 \in B_2, \cdots X_k \in B_k, X_{k+1} \leq x_{k+1}, \cdots, X_n \leq x_n)$$
$$= \prod_{i=1}^{k} P(X_i \in B_i) \prod_{i=k+1}^{n} P(X_i \leq x_i)$$

ASSERTION(0) is what we are assuming, and ASSERTION(n) is what we need to prove. By induction, it suffices to show ASSERTION(k) implies ASSERTION(k+1).

Fix reals $x_1, x_2, \cdots, x_k$ and Borel sets $B_{k+2}, \cdots, B_n$. Define two finite measures Q_1 and Q_2 on the reals by

$$\begin{aligned} Q_1(B) &= P(X_{k+1} \in B, X_1 \in B_1, X_2 \in B_2, \cdots X_k \in B_k, X_{k+2} \leq x_{k+2}, \cdots, X_n \leq x_n) \\ Q_2(B) &= P(X_k \in B) \prod_{i=1}^{k} P(X_i \in B_i) \prod_{i=k+2}^{n} P(X_i \leq x_i) \end{aligned}$$

If B is of the form $(-\infty, x]$, then by ASSERTION(k) we have $Q_1(B) = Q_2(B)$. The collection of sets $(-\infty, x]$ is closed under finite intersections and generates the Borel sets, so by the uniquesness of measure theorem, Q_1 and Q_2 agree on all Borel sets. This is ASSERTION(k+1).

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2. Let X be a RV. Let $p \geq 1$. Prove that

$$E|X|^p = \int_0^\infty px^{p-1} P(|X| \geq x) dx$$

Hint: Write $P(|X| \geq x)$ as the expected value of an indicator function. Then you should start seeing double (integrals that is).

SOLUTION: We have $P(|X| \geq x) = E[1_{|X| \geq x}]$. So

$$\int_0^\infty px^{p-1} P(|X| \geq x) dx = \int_0^\infty px^{p-1} E[1_{|X| \geq x}] dx$$

Consider $px^{p-1} 1_{|X(\omega)| \geq x}$ as a function on the product space $\Omega \times \mathbb{R}$. It is non-negative and measureable, so its integral with respect to the product measure $dP \times dx$ exists. So by Fubini (or Tonelli) the above iterated integral is equal to the interated integral in the other order. So the above is

$$= E[\int_0^\infty px^{p-1} 1_{|X| \geq x} dx] = E[\int_0^{|X|} px^{p-1} dx] = E[|X|^p]$$

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4. Let X_n be a sequence of random variables which converges to the random variable X a.s. Suppose there is a $p > 1$ such that

$$\sup_n E[|X_n|^p] < \infty$$

Prove that $E[|X_n - X|]$ converges to 0.

SOLUTION

By the uniform integrability condition, if we can show that $\{X_n\}$ is uniformly integrable this will show $E[|X_n - X|]$ converges to 0. (We also need to show that for each n, $E[|X_n|] < \infty$, but this follows easily if we can show the sequence is uniformly integrable.) So we need to show

$$\lim_{c \to \infty} \sup_n E[|X_n| 1_{|X_n| \geq c}] = 0$$

Since $1 - p < 0$ we have for $c \geq 1$,

$$|X_n| \, 1_{|X_n| \geq c} = |X_n|^p \, |X_n|^{1-p} 1_{|X_n| \geq c} \leq |X_n|^p \, c^{1-p}$$

So

$$E[|X_n| \, 1_{|X_n| \geq c}] \leq c^{1-p} \, E[|X_n|^p]$$

Thus

$$\sup_n E[|X_n| \, 1_{|X_n| \geq c}] \leq c^{1-p} \, M$$

where

$$M = \sup_n E[|X_n|^p] < \infty$$

Since $1 - p < 0$, c^{1-p} goes to zero as $c \to \infty$.

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6. We flip a fair coin infinitely many times. The sample space Ω consists of all sequences of 0's and 1's, 0 representing tails, 1 heads. We denote the elements of Ω by $(\omega_1, \omega_2, \omega_3, \cdots)$. We define a σ-field $\mathcal{F}$ as we did in class. Let n be a positive integer, and for $i = 1, 2, \cdots, n$ let ϵ_i be 0 or 1. Then define

$$E^n_{\epsilon_1, \cdots, \epsilon_n} = \{(\omega_1, \omega_2, \omega_3, \cdots) : \omega_i = \epsilon_i, \quad i = 1, 2, \cdots, n\}$$

Then $\mathcal{F}$ is the σ-field generated by all the $E^n_{\epsilon_1, \cdots, \epsilon_n}$. We proved in class that there is a probability measure P on $(\Omega, \mathcal{F})$ such that $P(E^n_{\epsilon_1, \cdots, \epsilon_n}) = 2^{-n}$. Let X_n be the random variable

$$X_n(\omega_1, \omega_2, \cdots) = \omega_n$$

(So X_n is 0 or 1 depending on the nth flip.) Show that $\{X_n\}_{n=1}^{\infty}$ is independent. (This is intuitively obvious. The point of the problem is to show it using the definition of independence of RV's.)
SOLUTION: By definition, the sequence of random variables is independent if the collection of sigma fields $\sigma(X_n)$ is independent. The key thing to observe is that since the random variable X_n only takes on the values 0 and 1, there are only four events in $\sigma(X_n)$: $\emptyset$, Ω, A_n^0, A_n^1, where

$$A_n^x = \{(\omega_1, \omega_2, \cdots) : \omega_n = x\}$$

for $x = 0, 1$. To show the collection of sigma fields is independent we must show that for a finite subset K of the natural numbers and $A_n \in \sigma(X_n)$ for $n \in K$ we have

$$P(\cap_{n \in K} A_n) \;=\; \prod_{n \in K} P(A_n)$$

If there is an A_n which is empty, then both sides are zero.

Now suppose all the A_n are non empty. If some A_k is Ω then we can leave it out of the $\cap_{n \in K} A_n$ without changing this intersection. And $P(A_k) = 1$, so we can leave this factor out of the product on the right. So we can assume that none of the A_n are Ω. So each A_n is either A_n^0 or A_n^1. So we are left with showing

$$P(\cap_{n \in K} A_n^{x_n}) = \prod_{n \in K} P(A_n^{x_n})$$

where the x_n are 0 or 1. Note that the sets in the above need not be of the form $E^n_{\epsilon_1, \cdots, \epsilon_n}$. But we can write them as disjoint unions of such sets and so compute their probabilities. We find of course that both sides of the above equation are $2^{-|K|}$ where $|K|$ is the cardinality of K.