Math 563 - Homework 5

1. Suppose X_n are real valued random variables defined on the same probability space. Prove that if X_n converges to 0 in probability, then there is a subsequence that converges to 0 a.s. Hint: show there is a subsequence n_k such that $P(|X_{n_k}| > 1/k) < 1/k^2$.

2. (from Durrett) Let X_n be an i.i.d. sequence of non-negative random variables that represent the lifetimes of a sequence of identical light bulbs. Let Y_n be another i.i.d. sequence of non-negative random variables. Y_n is the time we must wait after the nth bulb burns out before it is replaced. (We also assume $\{X_n, Y_n : n = 1, 2, 3, \cdots\}$ is independent.) Assume that EX_1 and EY_1 are both finite. Let W_t be the amount of time in $[0, t]$ that we have a working light bulb. Prove that

$$\frac{W_t}{t} \to \frac{E[X_1]}{E[X_1] + E[Y_1]} \quad a.s.$$

3. Let X_n be an independent sequence of real-valued random variables. Let $\mathcal{T}$ be their tail field.
(a) Prove that if a random variable is measurable with respect to $\mathcal{T}$, then it is equal to a constant a.s.
(b) Let $S_n = \sum_{k=1}^{n} X_k$. Prove that

$$\liminf_{n\to\infty} \frac{S_n}{n} \quad and \quad \limsup_{n\to\infty} \frac{S_n}{n}$$

are equal to a constant a.s.

4. Let X_n be an i.i.d. sequence of random variables with $P(X_n = +1) = P(X_n = -1) = 1/2$. Let c_n be a sequence of constants. Find a necessary and sufficient condition on the c_n for $\sum_n c_n X_n$ to converge a.s.

5. (from Durrett) Let X_n be an independent, identically distributed sequence of real-valued random variables. The radius of convergence of the power series $\sum_n X_n z^n$ is

$$\sup\{c \geq 0 : \sum_{n=1}^{\infty} |X_n| c^n < \infty\}$$

Note that it is a random variable. Define $\ln^+(x) = \max\{\ln(x), 0\}$. Prove that if $E[\ln^+(|X_1|)] < \infty$ then the radius of convergence is 1 a.s, and if this integral is infinite, then the radius of convergence is 0 a.s.

6. Let X be a real valued random variable. Let $\beta(t)$ be its characteristic function.

(a) Show that if there is an $a \neq 0$ such that $\beta(2\pi a) = 1$, then aX takes values in the integers a.s.
(b) Show that if there is an $a \neq 0$ such that $|\beta(2\pi a)| = 1$, then there is a real constant b such that $aX + b$ takes values in the integers a.s.
(c) Show that if there is an interval (a, b) such that $|\beta(t)| = 1$ for $t \in (a, b)$, then X is a constant a.s.