Math 563 - Homework 5

1. Suppose X_n are real valued random variables defined on the same probability space. Prove that if X_n converges to 0 in probability, then there is a subsequence that converges to 0 a.s. Hint: show there is a subsequence n_k such that $P(|X_{n_k}| > 1/k) < 1/k^2$.

Solution: By the definition of convergence in probability, for each k there is an integer n_k such that $n \geq n_k$ implies $P(|X_n| > 1/k) < 1/k^2$. We can also assume $n_k < n_{k+1}$. So this gives a subsequence n_k such that $P(|X_{n_k}| > 1/k) < 1/k^2$. Since $\sum_k k^{-2} < \infty$, this implies

$$\sum_k P(|X_{n_k}| > 1/k) < \infty$$

Let E be the event $|X_{n_k}| > 1/k \quad i.o.$. By the easy half of Borel Cantelli, the above implies $P(E) = 0$. If $\omega \notin E$, then $|X_{n_k}(\omega)| > 1/k$ happens for only a finite number of k. So there is an N such that $k > N$ implies $|X_{n_k}(\omega)| \leq 1/k$. This implies $X_{n_k}(\omega) \to 0$. So $X_{n_k} \to 0$ a.s.

2. (from Durrett) Let X_n be an i.i.d. sequence of non-negative random variables that represent the lifetimes of a sequence of identical light bulbs. Let Y_n be another i.i.d. sequence of non-negative random variables. Y_n is the time we must wait after the nth bulb burns out before it is replaced. (We also assume $\{X_n, Y_n : n = 1, 2, 3, \cdots\}$ is independent.) Assume that EX_1 and EY_1 are both finite. Let W_t be the amount of time in $[0, t]$ that we have a working light bulb. Prove that

$$\frac{W_t}{t} \to \frac{E[X_1]}{E[X_1] + E[Y_1]} \quad a.s.$$

3. Let X_n be an independent sequence of real-valued random variables. Let $\mathcal{T}$ be their tail field.
(a) Prove that if a random variable is measurable with respect to $\mathcal{T}$, then it is equal to a constant a.s.
(b) Let $S_n = \sum_{k=1}^n X_k$. Prove that

$$\liminf_{n\to\infty} \frac{S_n}{n} \quad and \quad \limsup_{n\to\infty} \frac{S_n}{n}$$

are equal to a constant a.s.

Solution: (a) If X is measurable with respect to $\mathcal{T}$, then for all real x, the event $\{X \leq x\}$ is in $\mathcal{T}$. So by the Kolmogorov 0-1 law, its probability is 0 or 1. So for all x, $F(x) = 0$ or 1. Let $c = \sup\{x : F(x) = 0\}$. Note that $c = \infty$ would contradict $\lim_{x\to\infty} F(x) = 1$. And $c = -\infty$ would contradict $\lim_{x\to-\infty} F(x) = 0$. So c is finite and since $F(x)$ is increasing, $F(x) = 0$ for $x < c$ and $F(x) = 1$ for $x > c$. This implies $X = c$ a.s.

(b) By part (a), it suffices to show that these random variables are measurable with respect to the tail field. We consider only the first random variable. We must show it is measurable with respect to $\sigma(X_m, X_{m+1}, \cdots)$ for all m. Fix an m. Then $S_n/m \to 0$ as $n \to \infty$. So

$$\liminf_{n\to\infty} \frac{S_n}{n} = \liminf_{n\to\infty} \frac{S_n - S_m}{n}$$

$S_n - S_m$ is the sum of X_k for $k = m+1, \cdots n$, so it is measurable with respect to $\sigma(X_m, X_{m+1}, \cdots)$. Measurability is preserved under countable inf's and limits, so $\liminf_{n\to\infty}(S_n - S_m)/n$ is measurable with respect to $\sigma(X_m, X_{m+1}, \cdots)$.

4. Let X_n be an i.i.d. sequence of random variables with $P(X_n = +1) = P(X_n = -1) = 1/2$. Let c_n be a sequence of constants. Find a necessary and sufficient condition on the c_n for $\sum_n c_n X_n$ to converge a.s.

Solution: Since $|c_n X_n| = |c_n|$, the series can only converge if c_n converges to zero. So we assume this is the case. In particular this implies $c = \sup_n |c_n|$ is finite. Now we apply the Kolmogorov three series theorem with $b > c$. So $Y_n = X_n 1(|X_n| > b)$ is just X_n. So the first two of the three series are trivial. So the theorem says $\sum_n c_n X_n$ converges a.s. if and only if

$$\sum_n var(c_n X_n) < \infty$$

We have

$$\sum_n var(c_n X_n) = \sum_n c_n^2 var(X_n) = \sum_n c_n^2$$

Thus $\sum_n c_n X_n$ converges a.s. if and only if $\sum_n c_n^2 < \infty$.

5. (from Durrett) Let X_n be an independent, identically distributed sequence of real-valued random variables. The radius of convergence of the power series

$\sum_n X_n z^n$ is

$$\sup\{c \geq 0 : \sum_{n=1}^{\infty} |X_n| c^n < \infty\}$$

Note that it is a random variable. Define $\ln^+(x) = \max\{\ln(x), 0\}$. Prove that if $E[\ln^+(|X_1|)] < \infty$ then the radius of convergence is 1 a.s, and if this integral is infinite, then the radius of convergence is 0 a.s.

Solution: We won't need this fact, but it is worth noting that by problem 3.a., the radius of convergence is equal to a constant a.s.

First suppose that $E[\ln^+(|X_1|)] < \infty$. Let $\delta > 0$. Then $E[\ln^+(|X_1|)/\delta] < \infty$. By a previous homework problem,

$$E[\ln^+(|X_1|)/\delta] = \int_0^{\infty} P(\ln^+(|X_1|)/\delta \geq t) dt$$

By the usual argument (left hand Riemann sums are a lower bound on the integral of a decreasing function,

$$\sum_{n=1}^{\infty} P(\ln^+(|X_1|)/\delta \geq n) < \infty$$

Since the X_n are identically distributed,

$$\sum_{n=1}^{\infty} P(\ln^+(|X_n|)/\delta \geq n) < \infty$$

i.e.,

$$\sum_{n=1}^{\infty} P(|X_n| \geq \exp(\delta n)) < \infty$$

By the easy half of the Borel-Cantelli lemma,

$$P(|X_n| \geq \exp(\delta n) \quad i.o.) = 0$$

and so

$$P(|X_n| \geq \exp(\delta n) \quad for \quad finite \quad number \quad of \quad n) = 1$$

Consider an ω such that $|X_n(\omega)| \geq \exp(\delta n)$ for only finitely many n. Then there is an N, depending on ω, such that $|X_n(\omega)| < \exp(\delta n)$ for $n \geq N$. This implies $\sum_n c^n |X_n(\omega)|$ converges if $c < \exp(-\delta)$. This is true for all positive δ, so taking a countable sequence of δ converging to 0, we have shown the radius of converges is at least 1 a.s. If it is greater than 1 with nonzero probability, then $\sum_n |X_n|$ converges with nonzero probability and hence with probability 1. So $X_n \to 0$ a.s. This implies X_n converges to zero in probability and this contradicts the hypothesis that the X_n are identically distributed.

Now suppose that $E[\ln^+(|X_1|)] = \infty$. Then for all positive δ,

$$\sum_{n=1}^{\infty} P(\ln^+(|X_n|)/\delta \geq n) = \infty$$

By the hard half of the Borel-Cantelli lemma, $P(\ln^+(|X_n|)/\delta \geq n \quad i.o.) = 1$. So $P(|X_n| \geq \exp(n\delta) \quad i.o.) = 1$. If $|X_n| \geq \exp(n\delta)$ infinitely often, then $|X_n| \exp(-n\delta)$ does not converge to zero. So

$$\sum_n |X_n| \exp(-n\delta)$$

does not converge. So the radius of convergence is at most $\exp(-\delta)$. This is true for all $\delta > 0$, so taking δ to ∞ through a countable sequence, we conclude the radius is 0 a.s.

6. Let X be a real valued random variable. Let $\beta(t)$ be its characteristic function.

(a) Show that if there is an $a \neq 0$ such that $\beta(2\pi a) = 1$, then aX takes values in the integers a.s.
(b) Show that if there is an $a \neq 0$ such that $|\beta(2\pi a)| = 1$, then there is a real constant b such that $aX + b$ takes values in the integers a.s.
(c) Show that if there is an interval (a, b) such that $|\beta(t)| = 1$ for $t \in (a, b)$, then X is a constant a.s.

Solution:
(a) $\beta(2\pi a) = 1$ implies $E[\exp(2\pi i a X)] = 1$. So $E[1 - \exp(2\pi i a X)] = 0$. Taking the real part, $E[1 - \cos(2\pi a X)] = 0$. Since $1 - \cos(2\pi a X) \geq 0$, this implies $1 - \cos(2\pi a X) = 0$ a.s. So aX is an integer a.s.

(b) If $|\beta(2\pi a)| = 1$, then there is a real θ such that $\beta(2\pi a) = \exp(i\theta)$. So $E[1 - \exp(2\pi i a X - i\theta)] = 0$. Taking the real part, $E[1 - \cos(2\pi a X - \theta)] = 0$. So $aX - \theta/(2\pi)$ is an integer a.s.

(c) By part (b), for $t \in (a,b)$ there is a real number $c(t)$ such that $2\pi tX - c(t)$ is an integer a.s. Note that the execeptional set in this “a.s.” can depend on t. So we consider only the rational t in (a,b). Then there is a set E with $P(E) = 1$ such that $2\pi tX(\omega) - c(t)$ is an integer for $\omega \in E$. If X is not a constant on E, then we can pick $\omega_1, \omega_2 \in E$ with $X(\omega_1) \neq X(\omega_2)$. Since the difference of two integers is an integer, for all rational t in (a,b), $2\pi t[X(\omega_1) - X(\omega_2)]$ is an integer. This is impossible.