Math 563 - Homework 6

1. (from Durrett) Let X_n be a sequence of integer valued random variables, X another integer valued random variable. Prove that X_n converge to X in distribution if and only if

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

for all integers m.

2. Suppose that the random variables X_n are defined on the same probability space and there is a constant c such that X_n converges in distribution to the random variable c. Prove or disprove each of the following
(a) X_n converges to c in probability
(b) X_n converges to c a.s.

3. Let X be a real valued random variable with characteristic function $\beta(t)$. Suppose that $E[|X|^n] < \infty$ for some positive integer n. Prove that the nth derivative of $\beta(t)$ exists and $E[X^n] = (-i)^n \beta^{(n)}(0)$. Hint: the bound $|e^{i\theta} - 1| \leq |\theta|$ for real θ is useful. If you get really stuck, you can find the proof on p. 223 of the text.

4. Let μ_n be a sequence of probability measures which have densities $f_n(x)$ with respect to Lebesgue measure. Suppose that $f_n(x) \to f(x)$ a.e. where $f(x)$ is a density, i.e., a non-negative function with integral 1. Prove that μ_n converges in distribution to μ where μ is $f(x)$ time Lebesgue measure.

5. Let X_n be an i.i.d. sequence with $EX_n = 0$. Define $S_n = X_1 + \cdots + X_n$. Prove that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty \quad a.s.$$

Hints: central limit theorem and Kolmogorov zero-one law.

6. ("Self-normalized sums" from Durrett) Let X_n be an i.i.d. sequence with $EX_n = 0$ and $E[X_n^2] = \sigma^2 < \infty$. Prove that

$$\frac{\sum_{k=1}^n X_k}{[\sum_{k=1}^n X_k^2]^{1/2}}$$

converges in distribution to the standard normal distribution (standard means the mean is zero, the variance is one).