Math 563 - Homework 6 Solutions

1. (from Durrett) Let X_n be a sequence of integer valued random variables, X another integer valued random variable. Prove that X_n converge to X in distribution if and only if

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

for all integers m.

Solution: First suppose that X_n converges to X in distribution. Since X is integer valued, its distribution function is continous at all x which are not integers. So for x not an integer,

$$\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$$

Subtract this equation for $x = m - 1/2$ from this equation for $x = m + 1/2$ and we get

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

Now assume that for all m,

$$\lim_{n\to\infty} P(X_n = m) = P(X = m)$$

We must show that for all m,

$$\lim_{n\to\infty} P(X_n \le m) = P(X \le m)$$

(Since all the RV's are integer valued, it suffices to show this for integer m.) For all finite $l < k$, $P(l \le X_n \le k)$ is a finite sum of $P(X_n = m)$, so by our assumption

$$\lim_{n\to\infty} P(l \le X_n \le k) = P(l \le X \le k) \tag{1}$$

Let $\epsilon > 0$. Since

$$\sum_{j=-\infty}^{\infty} P(X = j) = 1$$

we can find l and k such that

$$\sum_{j=l}^{k} P(X=j) \geq 1-\epsilon$$

So by (1) we can find an N such that $n \geq N$ implies

$$\sum_{j=l}^{k} P(X_n=j) \geq 1-2\epsilon$$

In particular, $n \geq N$ implies

$$P(X_n < l) < 2\epsilon, \quad P(X_n > k) < 2\epsilon$$

Note that we also have

$$P(X < l) < 2\epsilon, \quad P(X > k) < 2\epsilon$$

Now suppose m is between l and k. (We can increase k and decrease l to make this be the case.) Then

$$\begin{aligned} P(X_n \leq m) &= P(X_n < l) + P(l \leq X_n \leq m), \\ P(X \leq m) &= P(X < l) + P(l \leq X \leq m) \end{aligned}$$

By (1), there is an N' such that $n \geq N'$ implies

$$|P(l \leq X_n \leq m) - P(l \leq X \leq m)| < \epsilon$$

So if $n \geq \max\{N, N'\}$, then

$$\begin{aligned} &|P(X_n \leq m) - P(X \leq m)| \leq P(X_n < l) + P(X < l) \\ &+|P(l \leq X_n \leq m) - P(l \leq X \leq m)| \leq 5\epsilon \end{aligned}$$

2. Suppose that the random variables X_n are defined on the same probability space and there is a constant c such that X_n converges in distribution to the random variable c. Prove or disprove each of the following
(a) X_n converges to c in probability
(b) X_n converges to c a.s.

Solution:

(a) TRUE:

Let $\epsilon > 0$. We must show $P(|X_n - c| \geq \epsilon)$ converges to zero as $n \to \infty$.

$$\begin{aligned} & P(|X_n - c| \geq \epsilon) = P(X_n \leq c - \epsilon) + P(X_n \geq c + \epsilon) \\ \leq \quad & P(X_n \leq c - \epsilon) + P(X_n > c + \epsilon/2) = F_n(c - \epsilon) + 1 - F_n(c + \epsilon/2) \end{aligned}$$

where F_n is the distribution function of X_n. Note that the distribution function of the constant c is continuous everywhere except at c. So

$$\lim_{n\to\infty} F_n(c + \epsilon/2) = 1, \quad \lim_{n\to\infty} F_n(c - \epsilon) = 0$$

So the above converges to zero.

(b) FALSE: In class we constructed an example of a sequence X_n on $[0, 1]$ that converges to 0 in probability but does not converge to 0 a.s. Convergence in probability implies convergence in distribution, so this gives an example that converges in distribution to a constant but does converge to it a.s.

3. Let X be a real valued random variable with characteristic function $\beta(t)$. Suppose that $E[|X|^n] < \infty$ for some positive integer n. Prove that the nth derivative of $\beta(t)$ exists and $E[X^n] = (-i)^n \beta^{(n)}(0)$. Hint: the bound $|e^{i\theta} - 1| \leq |\theta|$ for real θ is useful. If you get really stuck, you can find the proof on p. 223 of the text.

Solution: See the book.

4. Let μ_n be a sequence of probability measures which have densities $f_n(x)$ with respect to Lebesgue measure. Suppose that $f_n(x) \to f(x)$ a.e. where $f(x)$ is a density, i.e., a non-negative function with integral 1. Prove that μ_n converges in distribution to μ where μ is $f(x)$ time Lebesgue measure.

Rant: Lots of people tried to do this using the dominated convergence theorem. I don't see any way to do it with DCT.

Solution: We proved several equivalent forms of convergence in distribution. One of them is that μ_n converges to μ in distribution if and only if for every open set O, $\liminf_{n\to\infty} \mu_n(O) \geq \mu(O)$. By Fatou's lemma,

$$\liminf_{n\to\infty} \mu_n(O) = \liminf_{n\to\infty} \int_O f_n(x)\, dx \geq \int_O \liminf_{n\to\infty} f_n(x)\, dx \ = \int_O f(x)\, dx = \mu(O)$$

5. Let X_n be an i.i.d. sequence with $EX_n = 0$ and $EX_n^2 < \infty$. Define $S_n = X_1 + \cdots + X_n$. Prove that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty \quad a.s.$$

Hints: central limit theorem and Kolmogorov zero-one law.

Solution (after Daniel): Let c be a finite constant. If $\sup_{m\geq n} S_m/\sqrt{m} \geq c$ for all n, then $\limsup_{n\to\infty} S_n/\sqrt{n} \geq c$. So

$$P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq c) \geq P(\cap_{n=1}^{\infty}\{\sup_{m\geq n} \frac{S_m}{\sqrt{m}} \geq c\})$$

By continuity of probability the right side equals

$$= \lim_{n\to\infty} P(\sup_{m\geq n} \frac{S_m}{\sqrt{m}} \geq c)$$

Clearly

$$\{\frac{S_n}{\sqrt{n}} \geq c\} \subset \{\sup_{m\geq n} \frac{S_m}{\sqrt{m}} \geq c\}$$

Hence

$$\lim_{n\to\infty} P(\sup_{m\geq n} \frac{S_m}{\sqrt{m}} \geq c) \geq \lim_{n\to\infty} P(\frac{S_n}{\sqrt{n}} \geq c)$$

By the central limit theorem this limit is $P(Z \geq c)$ where Z is a mean zero random variable with the same variance as X_n. For any finite c, $P(Z \geq c) > 0$. Thus

$$P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq c) > 0$$

Since $\limsup S_n/\sqrt{n}$ is measurable with respect to the tail field, the above probability can only be 0 or 1. So it must be 1. Using continuity of P,

$$P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty) = \lim_{N\to\infty} P(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq N) = \lim_{N\to\infty} 1 = 1$$

6. ("Self-normalized sums" from Durrett) Let X_n be an i.i.d. sequence with $EX_n = 0$ and $E[X_n^2] = \sigma^2 < \infty$. Prove that

$$\frac{\sum_{k=1}^n X_k}{[\sum_{k=1}^n X_k^2]^{1/2}}$$

converges in distribution to the standard normal distribution (standard means the mean is zero, the variance is one).

Solution:

$$\frac{\sum_{k=1}^n X_k}{[\sum_{k=1}^n X_k^2]^{1/2}} = \frac{\frac{1}{\sqrt{n}}\sum_{k=1}^n X_k}{\left[\frac{1}{n}\sum_{k=1}^n X_k^2\right]^{1/2}}$$

Let

$$\begin{aligned} Z_n &= \frac{1}{\sqrt{n}}\sum_{k=1}^n X_k \\ Y_n &= \left[\frac{1}{n}\sum_{k=1}^n X_k^2\right]^{-1/2} \end{aligned}$$

By the strong law of large numbers, $Y_n \to 1/\sigma$ a.s. By the central limit theorem, Z_n conveges in distribution to Z where Z has a normal distribution with mean zero and variance σ^2.

To complete the proof we need to show Z_n converges to Z in distribution and Y_n converges to a constant c a.s. implies Z_nY_n converges to cZ in distribution. In fact, this statement is still true if we replace the a.s. convergence of Y_n to c with convergence in probability. Here is Alex's proof of this fact.

Claim 1: If $Y_n - Z_n$ converges to 0 in probability and Z_n converges to Z in distribution, then Y_n converges to Z in distribution. Proof of claim: Let $x \in \mathbb{R}$. Let $\epsilon > 0$ be such that F is continuous at $x + \epsilon$ and at $x - \epsilon$ where F is the distribution function of Z. Then

$$\begin{aligned} &P(Y_n \le x) = P(Y_n \le x, Z_n - Y_n < \epsilon) + P(Y_n \le x, Z_n - Y_n \ge \epsilon) \\ &\le P(Z_n \le x + \epsilon) + P(|Z_n - Y_n| \ge \epsilon) \end{aligned}$$

The last term converges to 0 as $n \to \infty$. The other term converges to $F(x+\epsilon)$. Thus

$$\limsup_{n\to\infty} P(Y_n \le x) \le F(x + \epsilon)$$

Now

$$P(Z_n \le x - \epsilon) = P(Z_n \le x - \epsilon, Y_n - Z_n < \epsilon) + P(Z_n \le x - \epsilon, Y_n - Z_n \ge \epsilon)$$
$$\le P(Y_n \le x) + P(|Z_n - Y_n| \ge \epsilon)$$

The last term goes to zero as $n \to \infty$, and $P(Z_n \le x - \epsilon)$ converges to $F(x - \epsilon)$. Thus

$$\liminf_{n\to\infty} P(Y_n \le x) \ge F(x - \epsilon)$$

Now suppose F is continous at x. Then using $\liminf_{n\to\infty} P(Y_n \le x) \le \limsup_{n\to\infty} P(Y_n \le x)$ we can let $\epsilon \to 0$ to conclude

$$\lim_{n\to\infty} P(Y_n \le x) = F(x)$$

Claim 2: If Z_n converges to Z in distribution and Y_n converges to 0 in probability, then $Y_n Z_n$ converges to 0 in probability. Proof of claim: Let $K > 0$.

$$P(|Y_n Z_n| \ge \epsilon) = P(|Y_n Z_n| \ge \epsilon, |Y_n| < \epsilon/K) + P(|Y_n Z_n| \ge \epsilon, |Y_n| \ge \epsilon/K)$$
$$\le P(|Z_n| \ge K) + P(|Y_n| \ge \epsilon/K)$$

The last term goes to zero as $n \to \infty$ since Y_n convergs to 0 in probability. So

$$\limsup_{n\to\infty} P(|Y_n Z_n| \ge \epsilon) \le \limsup_{n\to\infty} P(|Z_n| \ge K) = P(|Z| \ge K)$$

since Z_n converges to Z in distribution. As $K \to \infty$, $P(|Z| \ge K)$ goes to 0, so $\limsup_{n\to\infty} P(|Y_n Z_n| \ge \epsilon) = 0$, which implies $P(|Y_n Z_n| \ge \epsilon)$ converges to 0.

Claim 3: If Z_n converges to Z in distribution and Y_n converges to c in probabililty, then $Y_n Z_n$ converges to cZ in distribution. Proof of claim: Note that cZ_n converges to cZ in distribution. Note that Y_n converging to c in probability is equivalent to $Y_n - c$ converging to 0 in probability. By claim 2, $Y_n Z_n - cZ_n = (Y_n - c)Z_n$ converges to 0 in probability. By claim 1 this implies $Y_n Z_n = (Y_n Z_n - cZ_n) + cZ_n$ converges to cZ in distribution.

Here is a different proof that if Z_n converges to Z in distribution and Y_n converges to a constant c a.s., then $Z_n Y_n$ converges to cZ in distribution. By the continuity theorem it suffices to show that the characteristic function of $Z_n Y_n$ converges to that of cZ, i.e., for all t,

$$\lim_{n\to\infty} E[\exp(itY_n Z_n)] = E[\exp(itcZ)]$$

Let $M > 0$. By the triangle inequality,

$$|E[\exp(itY_n Z_n) - \exp(itcZ)]| \leq |T_1| + |T_2| + |T_3| + |T_4|$$

where

$$\begin{aligned} T_1 &= E[\exp(itY_n Z_n) - \exp(itY_n Z_n)1(|Z_n| \leq M)] \\ T_2 &= E[(\exp(itY_n Z_n) - \exp(iZ_n ct))1(|Z_n| \leq M)] \\ T_3 &= E[\exp(iZ_n ct)1(|Z_n| \leq M) - \exp(iZ_n ct)] \\ T_4 &= E[\exp(iZ_n ct) - \exp(iZct)] \end{aligned}$$

We have

$$T_1 = E[\exp(itY_n Z_n)1(|Z_n| > M)]$$

So

$$|T_1| \leq E[1(|Z_n| > M)] = P(|Z_n| > M)$$

As $n \to \infty$, this converges to $P(|Z| > M)$, assuming M is not a discontinuity point. The same argument applies to T_3.

If $|Z_n| \leq M$ and $Y_n \to c$, then $\exp(itY_n Z_n) - \exp(iZ_n ct)$ converges to zero. So the integrand in T_2 converges to zero pointwise a.s. It is dominated by 2, so by the DCT, $T_2 \to \infty$.

T_4 converges to zero since Z_n converges to Z in distribution. Thus we have shown

$$\limsup_{n\to\infty} |E[\exp(itY_n Z_n) - \exp(itcZ)]| \leq 2P(|Z| > M)$$

As $M \to \infty$, $P(|Z| > M)$ converges to zero, so this completes the proof.