Math 563 - Homework 7

1. The gamma distribution is a two parameter family of distributions for non-negative random variables. The parameters λ and γ are both positive and the density is

$$f(x) = \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\lambda x}, \quad x \geq 0$$

where Γ is the usual gamma function. ($\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$.)
(a) Compute the characterisitic function of this distribution.
(b) Show that the gamma distributions are infinitely divisible.

2. Suppose X is a bounded RV that is infinitely divisible. Show it is a constant. Hint: show its variance is zero.

3. The theorem that we proved in class on Poisson convergence (law of rare events) has the following generalization:
Theorem For each n, let $\{X_{n,k} : k = 1, 2, \cdots, n\}$ be independent random variables whose values are non-negative integers. Let $p_{n,k} = P(X_{n,k} = 1)$ and $\epsilon_{n,k} = P(X_{n,k} \geq 2)$. Suppose that there is a $\lambda \in (0, \infty)$ such that
(1) $\sum_{k=1}^n p_{n,k} \to \lambda$
(2) $\max_{1 \leq k \leq n} p_{n,k} \to 0$
(3) $\sum_{k=1}^n \epsilon_{n,k} \to 0$
Let $S_n = \sum_{k=1}^n X_{n,k}$. Then S_n converges in distribution to a Poisson distribution with mean λ.

For this problem do one of the following.
(a) Prove the above.
(b) Use the new theorem to show the following. Let N be an infinitely divisible random variable whose values are non-negative integers and such that

$$\frac{P(N=2)}{P(N=0)} = \frac{1}{2} \left(\frac{P(N=1)}{P(N=0)} \right)^2$$

Then N has a Poisson distribution. (Note that the above equation is satisfied by a Poisson RV. Both sides are $\lambda^2/2$.)

4. A random variable X is symmetric if μ_X is invariant with respect to the $x \to -x$. Equivalently, X is symmetric if X and $-X$ have the same distribution. Show that if X is a symmetric random variable with finite

variance which is infinitely divisible, then its characteristic function may be written as

$$\exp\left[2\int_{[0,\infty)} \frac{\cos(tx)-1}{x^2}\,\mu(dx)\right]$$

where μ is a finite measure on $[0,\infty)$ and $(\cos(tx)-1)/x^2$ is understood to be $-t^2/2$ at $x=0$.