Math 563 - Homework 7

1. The gamma distribution is a two parameter family of distributions for non-negative random variables. The parameters λ and γ are both positive and the density is

$$f(x) = \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\lambda x}, \quad x \geq 0$$

where Γ is the usual gamma function. ($\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$.)
(a) Compute the characterisitic function of this distribution.
(b) Show that the gamma distributions are infinitely divisible.

2. Suppose X is a bounded RV that is infinitely divisible. Show it is a constant. Hint: show its variance is zero.

Solution: Let M be such that $|X| \leq M$ a.s. Let n be a positive integer. Infinitely divisible means that there exists i.i.d. $X_1, X_2, \cdots, X_n$ such that $X_1 + X_2 + \cdots + X_n$ is equal in distribution to X.

If $X_k > M/n$ for all k, then $\sum_{k=1}^n X_k > M$. So

$$P(X_k > M/n, \forall k) \leq P(\sum_{k=1}^n X_k > M) = P(X > M) = 0$$

Since they are i.i.d., $P(X_k > M/n, \forall k) = P(X_1 > M/n)^n$.
So $P(X_1 > M/n) = 0$. A similar argument shows $P(X_1 < -M/n) = 0$. So $|X_1| \leq M/n$ a.s.

The above bound implies $var(X_1) \leq E[X^2] \leq M^2/n^2$. So

$$var(X) = var(\sum_{k=1}^n X_k) = \sum_{k=1}^n var(X_k) \leq n \frac{M^2}{n^2} = \frac{M^2}{n}$$

This is true for all n, so $var(X) = 0$. Hence $X = E[X]$ a.s.

Comment: The conclusion that $|X_k| \leq M/n$ does not just follow from $|X| \leq M$. Consider the following example. Take the probability space to be $\{1, 2, \cdots, n\}$ with the uniform probability measure, i.e., each point has probability $1/n$. Define the RV X_k to be 1 when $\omega = k$ and 0 when $\omega \neq k$. Then $X_1 + X_2 + \cdots + X_n = 1$. But we do not have $|X_k| \leq 1/n$. Note that in this example the X_k are identically distributed, but not independent.

3. The theorem that we proved in class on Poisson convergence (law of rare events) has the following generalization:

Theorem For each n, let $\{X_{n,k} : k = 1, 2, \cdots, n\}$ be independent random variables whose values are non-negative integers. Let $p_{n,k} = P(X_{n,k} = 1)$ and $\epsilon_{n,k} = P(X_{n,k} \geq 2)$. Suppose that there is a $\lambda \in (0, \infty)$ such that

(1) $\sum_{k=1}^{n} p_{n,k} \to \lambda$

(2) $\max_{1 \leq k \leq n} p_{n,k} \to 0$

(3) $\sum_{k=1}^{n} \epsilon_{n,k} \to 0$

Let $S_n = \sum_{k=1}^{n} X_{n,k}$. Then S_n converges in distribution to a Poisson distribution with mean λ.

For this problem do one of the following.

(a) Prove the above.

Solution: One approach is to adapt the proof of the theorem from class. Here is a shorter, less obvious proof. Define $X'_{n,k}$ to be 1 when $X_{n,k}$ is 1, 0 when $X_{n,k}$ is not 1. Let $S'_n = \sum_{k=1}^{n} X'_{n,k}$. Note that $P(X'_{n,k} = 1) = p_{n,k}$. By (1) and (2) and the theorem from class, S'_n converges to a Poisson RV with parameter λ.

Now

$$P(S'_n \neq S_n) \leq P(\cup_k \{X'_{n,k} \neq X_{n,k}\}) \leq \sum_{k=1}^{n} P(X'_{n,k} \neq X_{n,k}) = \sum_{k=1}^{n} \epsilon_{n,k} \to 0$$

as $n \to \infty$. So $S'_n - S_n$ converges to 0 in probability. It follows (proved in last homework) that $S_n + (S'_n - S_n) = S_n$ converges in distribution to the Poisson distribution.

(b) Use the new theorem to show that if N is a random variable whose values are non-negative integers and it is infinitely divisible, then it has a Poisson distribution.

Solution: This is not true without further conditions. We assume there is a $\lambda > 0$ such that $P(N = 0) = e^{-\lambda}$ and $P(N = 1) = \lambda e^{-\lambda}$. In particular, $P(N = 0) > 0$.

For each n let $X_{n,k}$, $k = 1, 2, \cdots, n$ be i.i.d. with $X_{n,1} + \cdots + X_{n,n} = N$ in distribution. We first argue that the $X_{n,k}$ can only take on nonnegative integer values. We have

$$0 = P(N < 0) = P(\sum_{k=1}^{n} X_{n,k} < 0) \geq P(X_{n,k} < 0, \forall k) = P(X_{n,1} < 0)^n$$

Hence $P(X_{n,k} < 0) = 0$. This implies that $P(N = 0) = P(X_{n,1} = 0)^n$. So $P(X_{n,1} = 0) > 0$. Now let m be a positive integer.
Consider $P(X_{n,k} \in (m-1, m))$. We have

$$\begin{aligned} &P(N \in (m-1,m)) \geq P(X_{n,1} \in (m-1,m), X_{n,k} = 0, k = 2, 3, \cdots) \\ &= P(X_{n,1} \in (m-1,m))P(X_{n,2} = 0)^{n-1} \end{aligned}$$

Since $P(N \in (m-1,m)) = 0$, we must have $P(X_{n,1} \in (m-1,m)) = 0$. This is true for all m. So $X_{n,1}$ only takes non-negative integer values.

Now let $p_n = P(X_{n,k} = 1)$, $\epsilon_n = P(X_{n,k} \geq 2)$. So $P(X_{n,k} = 0) = 1 - p_n - \epsilon_n$. By our assumptions

$$\begin{aligned} e^{-\lambda} &= P(N = 0) = P(X_{n,k} = 0, k = 1, 2, \cdots, n) = (1 - p_n - \epsilon_n)^n \\ \lambda e^{-\lambda} &= P(N = 1) = \sum_{j=1}^{n} P(X_{n,j} = 1, X_{n,k} = 0, k \neq j) = np_n(1 - p_n - \epsilon_n)^{n-1} \end{aligned}$$

The first equation implies that $p_n \to 0$ and $\epsilon_n \to 0$. Dividing the second equation by the first,

$$\lambda = \frac{np_n}{1 - p_n - \epsilon_n}$$

Hence $np_n \to \lambda$. This proves (1) and (2). Now take the ln of first equation:

$$\lambda = -n \ln(1 - p_n - \epsilon_n)$$

For $x \geq 0$, $-\ln(1-x) \geq x$, so

$$\lambda \geq n(p_n + \epsilon_n)$$

Since $np_n \to \lambda$, this implies $n\epsilon_n \to 0$ which proves (3).

4. A random variable X is symmetric if μ_X is invariant with respect to the $x \to -x$. Equivalently, X is symmetric if X and $-X$ have the same distribution. Show that if X is a symmetric random variable with finite variance which is infinitely divisible, then its characteristic function may be written as

$$\exp\left[2\int_{[0,\infty)} \frac{\cos(tx) - 1}{x^2}\,\mu(dx)\right]$$

where μ is a finite measure on $[0, \infty)$ and $(\cos(tx) - 1)/x^2$ is understood to be $-t^2/2$ at $x = 0$.

Solution: For any random varible,

$$\beta(-t) = E[\exp(-itX)] = \overline{E[\exp(itX)]} = \overline{\beta(t)}$$

If X is symmetric, then

$$\beta(-t) = E[\exp(-itX)] = E[\exp(itX)] = \beta(t)$$

Hence, $\beta(t) = \overline{\beta(t)}$. So $\beta(t)$ is real.

Since X is infintely divisible, by the Levy-Khinchin representation

$$\begin{aligned}
\beta(t) &= \exp\left(ibt + \int_{\mathbb{R}} \frac{e^{itx} - 1 - itx}{x^2}\, \mu(dx)\right) \\
&= \exp\left(\int_{\mathbb{R}} \frac{\cos(itx) - 1}{x^2}\, \mu(dx)\right) \exp\left(ibt + \int_{\mathbb{R}} \frac{i\sin(tx) - itx}{x^2}\, \mu(dx)\right)
\end{aligned}$$

This is real for all t, so $bt + \int_{\mathbb{R}} \frac{\sin(tx) - tx}{x^2}\, \mu(dx)$ is an integer multiple of 2π for all t. But it is continuous in t and equals 0 at $t = 0$. So this quantity is zero for all t.

Thus

$$\beta(t) = \exp\left(\int_{\mathbb{R}} \frac{\cos(itx) - 1}{x^2}\, \mu(dx)\right)$$

Let $p = \mu(0)$. Define μ^+ by $\mu^+(B) = \mu(B \cap (0, \infty)$. Define μ^- by $\mu^-(B) = \mu(B \cap (-\infty, 0)$. Then $\mu = \mu^+ + \mu^- + p\delta_0$, where δ_0 is unit mass at 0. For $B \subset [0, \infty)$, define $\nu(B) = \mu^+(B) + \mu^-(-B) + p\delta_0(B)$. Then

$$\int_{\mathbb{R}} \frac{\cos(itx) - 1}{x^2}\, \mu(dx) = \int_{[0,\infty)} \frac{\cos(itx) - 1}{x^2}\, \nu(dx)$$

Comment: If $\mu(\{0\}) > 0$, we do not have

$$\int_{\mathbb{R}} f(x)\mu(dx) = \int_0^\infty f(x)\mu(dx) + \int_{-\infty}^0 f(x)\mu(dx)$$

In fact, $\int_0^\infty f(x)\mu(dx)$ is ambiguous. It could mean $\int_{(0,\infty)} f(x)\mu(dx)$ or $\int_{[0,\infty)} f(x)\mu(dx)$, and they are not equal.