Math 563 - Take Home Midterm Solutions

1. Let $f(x)$ be a Borel measurable real-valued function on $[0,1]$ with $\int_0^1 f^2(x)\,dx < \infty$. Let $\{U_n\}_{n=1}^\infty$ be an independent sequence of random variables, each of which is uniformly distributed on $[0,1]$, i.e., the distribution of U_n is Lebesgue measure on $[0,1]$. Let

$$S_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i)$$

Prove that S_n converges to the constant $\int_0^1 f(x)\,dx$ in probability.

SOLUTION: By a theorem from class, the independence of the U_i implies that the sequence $f(U_i)$ is independent. By the law of the unconsious statistcian,

$$E[f^2(U_i)] = \int_0^1 f^2(x)\,dx < \infty$$

The finiteness of the second moment implies the first moment is finite and

$$E[f(U_i)] = \int_0^1 f(x)\,dx$$

It is easy to check that the $f(U_i)$ are identically distibuted since the U_i are. So by the weak law of large numbers, S_n converges in probability to the mean of $f(U_i)$, i.e., to the constant $\int_0^1 f(x)\,dx$.

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2. Let X_n be an independent, identically distributed sequence of non-negative random variables. Prove that $EX_1 < \infty$ if and only if $P(X_n \geq n\ i.o.) = 0$.

SOLUTION: By the Borel-Cantelli lemma, $P(X_n > n\ i.o.) = 0$ if and only if $\sum_n P(X_n > n) < \infty$. Since they are identically distributed, $P(X_n > n) = P(X_1 > n)$. So we have reduced the problem to showing that $EX_1 < \infty$ if and only $\sum_n P(X_1 > n) < \infty$. By a homework problem,

$$E[X_1] = \int_0^\infty P(X_1 \geq x)dx$$

Note that $P(X_1 \geq x)$ is a decreasing function of x. So

$$\sum_{n=1}^\infty P(X_1 \geq n) \leq \int_0^\infty P(X_1 \geq x) \leq \sum_{n=0}^\infty P(X_1 \geq n)$$

which proves the needed result.

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3. Let A_n be a sequence of events which converges to the event A in the following sense. If $\omega \in A$, then there is an N such that that $n \geq N$ implies $\omega \in A_n$. If $\omega \notin A$, then there is an N such that that $n \geq N$ implies $\omega \notin A_n$. Prove that $P(A_n)$ converges to $P(A)$.

SOLUTION: If $1_A(\omega) = 1$, then there is an N such that that $n \geq N$ implies $1_{A_n}(\omega) = 1$. And if $1_A(\omega) = 0$, then there is an N such that that $n \geq N$ implies $1_{A_n}(\omega) = 0$. Thus 1_{A_n} converges to 1_A pointwise for all ω. We have $|1_{A_n}| \leq 1$ and $E[1] < \infty$. So by the dominated convergence theorem, $E[1_{A_n}]$ converges to $E[1_A]$. Since $E[1_{A_n}] = P(A_n)$ and $E[1_A] = P(A)$, this proves the result.

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4. Let A be an event in $(\Omega, \mathcal{F}, P)$ with $P(A) > 0$. Define a new probability measure on $(\Omega, \mathcal{F})$ by $Q(B) = P(A \cap B)/P(A)$. Let X be a nonnegative random variable which is independent of A, meaning that $\sigma(X)$ and $\sigma(A)$ are independent. ($\sigma(A)$ is just $\{A, A^c, \Omega, \emptyset\}$.) Prove that $\int X dP = \int X dQ$. Hint: start with a simple random variable X.

SOLUTION: Let Z be a simple random variable which is measurable with respect to $\sigma(X)$. (The hint notwithstanding, it is confusing to call the simple random variable X.) So

$$Z = \sum_{j=1}^{m} c_j 1_{E_j}$$

where the E_j are disjoint and in $\sigma(X)$. So E_j and A are independent. So $Q(E_j) = P(E_j \cap A)/P(A) = P(E_j)P(A)/P(A) = P(E_j)$. Thus $\int Z\, dP = \int Z\, dQ$.

Now we let X_n be a sequence of simple nonnegative random variables which increase to X pointwise and (this is crucial) are measurable with respect to $\sigma(X)$. In the standard construction of the X_n, the events involved in the definition of X_n are of the form $X^{-1}(B)$ where B is an interval or half-infinite interval. So these events are in $\sigma(X)$ and so these X_n are measurable with respect to $\sigma(X)$. So $\int X_n\, dP = \int X_n\, dQ$. Now let $n \to \infty$ and use the monotone convergence theorem on both sides to get $\int X\, dP = \int X\, dQ$.

Here is different solution due to Ben. By a homework problem,
$\int X\, dQ = \int_0^\infty Q(X \geq x) dx$. Now for all x, the event $X \geq x$ is in $\sigma(X)$ and so is independent of A. So
$Q(X \geq x) = P(\{X \geq x\} \cap A)/P(A) = P(\{X \geq x\})P(A)/P(A) = P(X \geq x)$

Thus $\int_0^\infty Q(X \geq x)dx = \int_0^\infty P(X \geq x)dx = \int X dP$, again using the homework problem.

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5. Let X and Y be independent real valued random variables. Suppose that $P(X = x) = 0$ for all real x.
(a) Prove that $P(X = Y) = 0$.
(b) Prove that $P(XY = 1) = 0$.

SOLUTION:
Since X and Y are independent, their joint distribution is $\mu_X \times \mu_Y$. So

$$P(X = Y) = \int 1_A(x, y) d(\mu_X \times \mu_Y)$$

where A is the subset of the plane : $A = \{(x, y) : x = y\}$. So the indicator function $1_A(x, y)$ is 1 if and only if $x = y$. So we will write it as $1(x = y)$. Using Fubini (note the integrand is nonnegative),

$$\begin{aligned} \int 1(x = y) d(\mu_X \times \mu_Y) &= \int \left(\int 1(x = y)\, d\mu_X(dx) \right) d\mu_Y(dy) \\ &= \int P(X = y) d\mu_Y(dy) = \int 0\, d\mu_Y(dy) = 0 \end{aligned}$$

The proof of (b) is almost the same. Now $A = \{(x, y) : xy = 1\}$. So in place of $P(X = y)$ we get $P(X = 1/y)$, which is also zero.

Another approach is to write $P(X = Y)$ as $P(X + (-Y) = 0)$. Now X and $-Y$ are independent, so the distribution their sum is the convolution of their distributions ...

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6. Suppose X and Y are independent, identically distributed *non-negative* random variables. They have a density $f(x)$. Let $Z = X - Y$. The density of Z is $\exp(-|x|)/2$. (Z takes on all real values.) Find $f(x)$.

SOLUTION: Compute the characteristic function of Z :

$$\begin{aligned} \int e^{itx} \frac{1}{2} e^{-|x|}\, dx &= \frac{1}{2} \int_0^\infty e^{itx - x}\, dx + \frac{1}{2} \int_{-\infty}^0 e^{itx + x}\, dx \\ &= \frac{1}{2} \left(\frac{1}{1 - it} + \frac{1}{1 + it} \right) = \frac{1}{1 + t^2} \end{aligned}$$

Let $\beta(t)$ be the characteristic function of X and Y. Then the characteristic function of $-Y$ is $\beta(-t)$. So the characteristic function of $X - Y = X + (-Y)$ is $\beta(t)\beta(-t)$. We can factor $1/(1 + t^2)$ as the product of $1/(1 + it)$ and $1/(1 - it)$. Note that $1/(1 + it)$ is the characteristic function of an exponential distribution with mean 1. So we can take X and Y to have this distribution.