6. (from Resnick) Let $P$ be a probability measure on $\mathcal{B}(\mathbb{R})$, the Borel sets in $\mathbb{R}$. Prove that for any $E \in \mathcal{B}(\mathbb{R})$ and any $\epsilon > 0$ there exits a finite union of disjoint intervals $A$ such that $P(E \Delta A) < \epsilon$.

Hint: Define $\mathcal{F}$ to be the collection of Borel sets such that for all $\epsilon > 0$ there exists a finite union of disjoint intervals $A$ such that $P(E \Delta A) < \epsilon$. What can you prove about $\mathcal{F}$?

Solution: I am interpreting “intervals” to mean all intervals that are open, closed or half-open/half-closed, along with singletons, the entire real line and half-infinite intervals.

Clearly $\mathcal{F}$ contains all sets which are intervals. The $\sigma$-algebra generated by the intervals is the Borel sets. So if we can show that $\mathcal{F}$ is a $\sigma$-algebra, then $\mathcal{F}$ must be all the Borel sets.

Closed under complements: Let $E \in \mathcal{F}$. Let $\epsilon > 0$. Then there is a set $A$ which is the disjoint union of a finite collection of intervals such that $P(A \Delta E) < \epsilon$. The set $A^c$ is a finite disjoint union of intervals. And it is easy to check that $A \Delta E = A^c \Delta E^c$. So $P(A^c \Delta E^c) < \epsilon$. Thus $E^c \in \mathcal{F}$.

Closed under finite unions: By induction it is enough to show it is closed under the union of two sets. So let $E_1, E_2 \in \mathcal{F}$. Let $\epsilon > 0$. Then there are sets $A_1, A_2$ which are disjoint unions of intervals such that $P(E_i \Delta A_i) < \epsilon/2$ for $i = 1, 2$. Let $A = A_1 \cup A_2$. Note that $A$ is a finite disjoint union of intervals. It is easy to check that

$$A \Delta (E_1 \cup E_2) \subset (A_1 \Delta E_1) \cup (A_2 \Delta E_2)$$

So

$$P(A \Delta (E_1 \cup E_2)) \leq P((A_1 \Delta E_1) \cup (A_2 \Delta E_2)) \leq P(A_1 \Delta E_1) + P(A_2 \Delta E_2) < \epsilon/2 + \epsilon/2$$

Thus $E_1 \cup E_2 \in \mathcal{F}$.

Closed under countable unions: We have shown it is a field, so by the usual trick of writing a union as a disjoint union, it is enough to show it is closed under disjoint countable unions. So let $E_n \in \mathcal{F}$ be disjoint and let $E = \bigcup_{n=1}^{\infty} E_n$. Let $\epsilon > 0$. Since they are disjoint,

$$\sum_{n=1}^{\infty} P(E_n) \leq 1$$
So we can pick \( N \) so that

\[
\sum_{n=N+1}^{\infty} P(E_n) < \epsilon/2
\]

We have \( \bigcup_{n=1}^N E_n \in \mathcal{F} \). So there is a set \( A \) which is a finite disjoint union of intervals such that \( P((\bigcup_{n=1}^N E_n) \Delta A) < \epsilon/2 \). We have

\[
E \Delta A \subset ((\bigcup_{n=1}^N E_n) \Delta A) \cup (\bigcup_{n=N+1}^\infty E_n)
\]

So

\[
P(E \Delta A) \leq P((\bigcup_{n=1}^N E_n) \Delta A) + P(\bigcup_{n=N+1}^\infty E_n) < \epsilon/2 + \epsilon/2
\]

Thus \( E \in \mathcal{F} \).