Math 563 - Fall ’15 - Homework 6 Solutions

1. (from Durrett) Let $X_n$ be a sequence of integer valued random variables, $X$ another integer valued random variable. Prove that $X_n$ converge to $X$ in distribution if and only if

$$\lim_{n \to \infty} P(X_n = m) = P(X = m)$$

for all integers $m$.

**Solution:** We prove that if $\lim_{n \to \infty} P(X_n = m) = P(X = m)$ then we have convergence in distribution. The other direction is easy.

For finite $a$ and $b$, $P(a \leq X_n \leq b)$ is a finite sum of $P(X_n = m)$. So the hypothesis immediately implies

$$\lim_n P(a \leq X_n \leq b) = P(a \leq X \leq b)$$

Now let $\epsilon > 0$. Pick $L$ such that

$$P(-L \leq X \leq L) \geq 1 - \epsilon$$

Note for later use that this implies $P(X < -L) < \epsilon$. Then pick $N$ so that for $n \geq N$ we have

$$P(-L \leq X_n \leq L) \geq 1 - 2\epsilon$$

So $P(X_n < -L) < 2\epsilon$ for $n \geq N$. The triangle inequality gives

$$|P(X_n \leq b) - P(X \leq b)| = |P(X_n < -L) + P(-L \leq X_n \leq b) - P(-L \leq X \leq b)| + P(X < -L)|$$

$$\leq |P(-L \leq X_n \leq b) - P(-L \leq X \leq b)| + 3\epsilon$$

Thus

$$\limsup_n |P(X_n \leq b) - P(X \leq b)| \leq 3\epsilon$$

This is true for all $\epsilon > 0$. So the lim sup above is 0, i.e., $P(X_n \leq b)$ converges to $P(X \leq b)$. So $X_n$ converges to $X$ in distribution.

3. (a) Let $\mu_n$ be a sequence of probability measures which have densities $f_n(x)$ with respect to Lebesgue measure. Suppose that $f_n(x) \to f(x)$ a.e. where
$f(x)$ is a density, i.e., a non-negative function with integral 1. Prove that $\mu_n$ converges in distribution to $\mu$ where $\mu$ is $f(x)$ times Lebesgue measure.

**Solution:** We first show that $f_n$ converges to $f$ in $L^1$. Since $f_n \geq 0$, $(f - f_n)^+ \leq f$. (Note that it is not true that $(f - f_n)^- \leq f$.) Since $f$ is integrable and $(f - f_n)^+$ converges to zero a.e., the dominated convergence theorem implies

$$
\lim_n \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx = 0
$$

Since $f_n$ and $f$ have integral 1,

$$
\int_{-\infty}^{\infty} (f(x) - f_n(x)) \, dx = 0
$$

So

$$
\int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx = \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx
$$

Since the left side converges to zero the right side does too. Finally, we have

$$
\int_{-\infty}^{\infty} |f(x) - f_n(x)| \, dx = \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx + \int_{-\infty}^{\infty} (f(x) - f_n(x))^+ \, dx \to 0
$$

Fix an $x$. Letting $F_n, F$ be the distribution functions of $X_n, X$,

$$
|F_n(x) - F(x)| = \left| \int_{-\infty}^{x} (f_n(u) - f(u)) \, du \right|
$$

$$
\leq \int_{-\infty}^{x} |f_n(u) - f(u)| \, du \leq \int_{-\infty}^{\infty} |f_n(u) - f(u)| \, du
$$

The last integral converges to zero, so $F_n(x)$ converges to $F(x)$ for all $x$.

4. Suppose that the random variables $X_n$ are defined on the same probability space and there is a constant $c$ such that $X_n$ converges in distribution to the random variable $c$. Prove or disprove each of the following

(a) $X_n$ converges to $c$ in probability

**Solution:** This is true. Let $F_n$ be the distribution function of $X_n$ and $F$ the distribution function of $c$. So $F(x)$ is 0 for $x < c$ and is 1 for $x \geq c$. So the only point where $F$ is not continuous is $c$. So $F_n(x)$ converges to $F(x)$ for
x \neq c. So for any \( \epsilon > 0 \), \( F_n(c+\epsilon) - F_n(c-\epsilon) \) converges to \( F(c+\epsilon) - F(c-\epsilon) = 1 \). Now

\[
F_n(c + \epsilon) - F_n(c - \epsilon) = P(c - \epsilon < X_n \leq c + \epsilon) \geq P(c - \epsilon < X_n < c + \epsilon)
\]

So \( P(|X_n - c| < \epsilon) \) converges to 1. This shows \( X_n \) converges to \( c \) in distribution.

(b) \( X_n \) converges to \( c \) a.s. \textbf{Solution:} This is false. By part (a), if \( X_n \) converges to \( c \) in distribution then it converges in probability. But there are sequences that converge to 0 in probability but do not converge a.s., e.g. the “typewriter sequence.”

5. (from Durrett, converging together lemma) Suppose \( X_n \Rightarrow X \) and \( Y_n \Rightarrow c \) where \( c \) is a constant. Prove that \( X_n + Y_n \Rightarrow X + c \). Note that this implies that if \( X_n \Rightarrow X \) and \( Y_n - X_n \Rightarrow 0 \), then \( Y_n \Rightarrow X \).

\textbf{Solution:} First note that by a previous problem, since \( Y_n \) converges in distribution to a constant \( c \), it converges in probability to this constant.

Let \( F_n \) be the distribution function of \( X_n \) and \( F \) the distribution function of \( X \). Fix an \( x \) and let \( \epsilon > 0 \). We find upper and lower bounds on \( P(X_n + Y_n \leq x) \). We get an upper bound as follows.

\[
P(X_n + Y_n \leq x) = P(X_n + Y_n \leq x, |Y_n - c| \leq \epsilon) + P(X_n + Y_n \leq x, |Y_n - c| > \epsilon)
\]

\[
\leq P(X_n \leq x - c + \epsilon) + P(|Y_n - c| > \epsilon)
\]

We know that \( P(|Y_n - c| > \epsilon) \) converges to 0 as \( n \to \infty \). If \( x - c + \epsilon \) is a continuity point of \( F \), then \( P(X_n \leq x - c + \epsilon) = F_n(x - c + \epsilon) \) converges to \( F(x - c + \epsilon) \). So

\[
\limsup_n P(X_n + Y_n \leq x) \leq F(x - c + \epsilon)
\]

for all \( \epsilon \) such that \( x - c + \epsilon \) is a continuity point of \( F \). We can find a sequence of such \( \epsilon \) which converge to 0, and \( F(x - c + \epsilon) \) converges to \( F(x - c) \) as \( \epsilon \to 0 \). So

\[
\limsup_n P(X_n + Y_n \leq x) \leq F(x - c)
\]

We get a lower bound as follows.

\[
P(X_n \leq x - c - \epsilon)
\]
\[ P(X_n \leq x - c - \epsilon, |Y_n - c| < \epsilon) + P(X_n \leq x - c - \epsilon, |Y_n - c| \geq \epsilon) \]
\[ \leq P(X_n + Y_n \leq x) + P(|Y_n - c| \geq \epsilon) \]

So if \( x - c - \epsilon \) is a continuity point of \( F \), then taking the lim inf we get

\[ \liminf_n P(X_n + Y_n \leq x) \geq F(x - c - \epsilon) \]

If \( x - c \) is a continuity point of \( F \), then letting \( \epsilon \to 0 \) we get

\[ \liminf_n P(X_n + Y_n \leq x) \geq F(x - c) \]

Our two bounds imply

\[ \lim_n P(X_n + Y_n \leq x) = F(x - c) \]

Since \( P(X + c \leq x) = P(X \leq x - c) = F(x-c) \), this shows \( X_n + Y_n \) converges to \( X + c \) in distribution.