Math 565b - Homework 1

1. (a) Let $c > 0$ and $B_t$ be a standard Brownian motion. Let

$$X_t = c B_{t/c^2}$$  \hspace{1cm} (1)

Show that $X_t$ is another standard Brownian motion.

(b) Let $B_t$ be a standard Brownian motion. Let $X_t = t B_{1/t}$. (Define $X_0 = 0$.) Prove that $X_t$ is another standard Brownian motion.

2. Let $B_t$ be a standard BM. Let

$$X_t = \exp(\alpha B_t - \frac{1}{2} \alpha^2 t)$$  \hspace{1cm} (2)

where $\alpha$ is a real number. Show that $X_t$ is a martingale.

3. We say that real valued random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if there is a positive definite $n$ by $n$ matrix $A_{ij}$ and $n$ real numbers $\mu_i$ such that the joint density of the RV’s is

$$c \exp\left[-\frac{1}{2} \sum_{i,j=1}^{n} A_{ij} (x_i - \mu_i)(x_j - \mu_j)\right]$$  \hspace{1cm} (3)

where $c$ is the constant that makes this a probability density.

(a) (easy) Let $M$ be an invertible real $n$ by $n$ matrix and let

$$Y_t = \sum_{j=1}^{n} M_{ij} X_j$$  \hspace{1cm} (4)

Show that if $X_1, \ldots, X_n$ are jointly Gaussian, then so are $Y_1, \ldots, Y_n$.

(b) Let $X_1, \ldots, X_n$ be jointly Gaussian. To keep things under control assume they all have mean zero. Let

$$C_{ij} = E[X_i X_j]$$  \hspace{1cm} (5)

Show that $C$ is $A^{-1}$ where the inverse is a matrix inverse. Hint: First prove this when $A$ is diagonal. Then use the fact that any positive definite matrix is diagonalizable.

4. A process $X_t$ is said to be Gaussian if for any choice of times $t_1, t_2, \ldots, t_n$, the RV’s $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are jointly Gaussian. Let $X_t$ be a process with
$X_0 = 0$. Use the results of the previous problem to prove that the following are equivalent:

(i) For $0 \leq s < t$, $X_t - X_s$ has normal distribution with mean 0 and variance $t - s$, and $X_t - X_s$ is independent of $\sigma(\{X_u : u \leq s\})$.

(ii) $X_t$ is a Gaussian process with mean 0 and covariance

$$E X_t X_s = \min\{t, s\}$$

(6)

5. Let $B_t$ be a standard Brownian motion. Define

$$X(\omega) = \int_0^1 B_s(\omega)^2 \, ds$$

(7)

Prove that $X(\omega)$ is a random variable (i.e., prove the function is measurable) and compute the first two moments of $X$.

6. Let $B_t$ be a standard Brownian motion. Fix a $p > 0$. For each $n = 1, 2, \cdots$, let $t_i = i/n$. Prove that

$$n^{p/2-1} \sum_{i=1}^n |B_{t_{i+1}} - B_{t_i}|^p$$

(8)

converges in probability to a constant $v_p$ as $n \to \infty$. (Hint: Use the scaling of BM and the weak law of large numbers.)

7. Let $B_t$ be a standard Brownian motion. Prove that with probability one,

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{t}} > 0$$

(9)

(In fact, the lim sup is $+\infty$ w.p.1.).

8. Use the result of problem 7 to prove that $B_t$ is recurrent, i.e., for any real $x$, the set $\{t : B_t = x\}$ is unbounded with probability one.