1. Recall that we defined a function \( x(t) \) on \( \mathbb{Q}^+ \) (the positive rationals) to be \textit{regularizable} if its left and right hand limits both exist and are finite. In class I stated but did not prove

**Proposition:** Let \( x(t) : \mathbb{Q}^+ \to \mathbb{R} \) be regularizable. Define

\[
y(t) = \lim_{q \to t^+} x(q)
\]

where \( q \) is restricted to rationals. Then \( y(t) \) is an R-function, i.e., it is right continuous and its left hand limits exist.

Prove the proposition.

2. On the set of integrable random variables define a metric by

\[
d(X, X') = \inf\{\epsilon : P(|X - X'| > \epsilon) < \epsilon\}
\]

Prove that a sequence \( X_n \) of random variables converges to a random variable \( X \) in probability if and only if \( d(X_n, X) \to 0 \).

3. Let \( B_t \) be standard Brownian motion. Prove that \( \{B_t^2 : 0 \leq 1\} \) is uniformly integrable. Hint: Use a martingale.

4. Let \( T \) be a stopping time. The \( \sigma \)-field \( \mathcal{F}_T \) is the collection of events \( A \) such that for all \( t \geq 0 \), \( A \cap \{T \leq t\} \) is in \( \mathcal{F}_t \). Suppose that \( \mathcal{F}_t \) is a right continuous filtration. Prove that if we replace \( \{T \leq t\} \) by \( \{T < t\} \) in this definition then we get the same \( \sigma \)-field \( \mathcal{F}_T \).

5. Let \( B_t \) be standard Brownian motion, and \( \mathcal{F}_t^B \) the filtration where \( \mathcal{F}_t^B \) is generated by \( \{B_s : 0 \leq s \leq t\} \). Recall that we proved that all the events in \( \mathcal{F}_t^B \) have probability 0 or 1. Let \( f(t) \) be a function on \((0, \infty)\) such that \( f(t) > 0 \) for all \( t > 0 \). Define a random variable \( X \) by

\[
X = \limsup_{t \to 0^+} \frac{B_t}{f(t)}
\]

The lim sup always exists, but of course it may be \( \infty \). Prove that the random variable \( X \) is a constant (possibly infinite).

6. Use the result from the previous problem to prove that

\[
\limsup_{t \to 0^+} \frac{B_t}{\sqrt{t}} = \infty
\]
with probability one. $B_t$ is still a standard Brownian motion.

7. (Watkins, p. 19) Let $B_t$ be standard Brownian motion. Let $a > 0$ and define $\tau = \inf\{t \geq 0 : |B_t| = a\}$ Using the martingale $B_t^2 - t$ you can show $E\tau = a^2$. Show that $E\tau^2 = 5a^2/3$. Hint: Show that $B_t^4 - 6tB_t^2 + 3t^2$ is a martingale. You can show it is a martingale by brute force, but a much shorter method is to use the exponential martingale.

8. (Watkins, p. 20) Let $N_t$ be a Poisson process. Let $\tau_n$ be the hitting time of $n$, i.e., $\tau = \inf\{t : N_t = n\}$. Let $\sigma_n = \tau_n - \tau_{n-1}$. Prove that $\sigma_n$ is an i.i.d. sequence with exponential distribution.

9. (Watkins, p. 21) State and prove a reflection principle for symmetric Levy processes and general stopping times. We stated it for Brownian motion at the start of the course. You can also find a statement for BM in Watkins notes on page 21.

10. (Durrett, p. 402) Show that

$$X_t = \frac{\exp(B_t^2/(1+t))}{\sqrt{1+t}}$$

(5)

is a martingale and use this to show

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log t}} \leq 1$$

(6)

with probability one.