Math 565b - Homework 4

1. Consider the semigroup

\[ T(t)f(x) = f(x + t) \]  

Find the generator of \( T(t) \). Take the Banach space to be \( C_0(\mathbb{R}) \).

2. Consider a generalized Poisson process with \( Y_k = \pm 1 \) with probability \( 1/2 \). This is like a Poisson process in which instead of always going up by 1 when it jumps, the process either goes up or down by 1 (with equal probability) when it jumps. Recall that the generalized Poisson process \( X_t \) is defined by

\[ X_t = \sum_{k=1}^{N_t} Y_k \]  

where \( Y_k \) is iid and \( N_t \) is a Poisson process with rate \( \lambda \).

(a) Find the generator of the semigroup associated with this process.
(b) The theory we have developed shows that this generator must be a dissipative operator. Prove this directly using your answer to (a).

3. Let \( X_t \) be a Brownian motion with \( EX_t = \nu t \) and \( \text{var}(X_t) = \sigma^2 t \) where \( \nu \) and \( \sigma^2 \) are constants. (Note that you can obtain \( X_t \) by taking \( B_t \) to be a standard Brownian motion and letting \( X_t = \sigma^2 B_t + \mu t \).) Find the generator of the semigroup associated with this Markov process.

4. Consider a Gaussian process with mean zero and covariance \( C(s, t) \). It is defined for \( t \geq 0 \). Show that the process is a Markov process if and only if the covariance satisfies:

\[ C(s, u)C(t, t) = C(s, t)C(t, u) \]  

for \( 0 \leq s < t < u \).

5. Let \( B_t \) be standard Brownian motion. Let \( T(t) \) be the associated semigroup and \( R_\lambda \) its resolvent. Show that the resolvent is an integral operator, i.e.,

\[ R_\lambda f(x) = \int_{-\infty}^{\infty} r_\lambda(x, y)f(y)dy \]  

and

\[ r_\lambda(x, y) = \frac{1}{\sqrt{2\lambda}} \exp(-\sqrt{2\lambda}|x - y|) \]
6. Let \( P(t, x, dy) \) be a time homogeneous transition function. In particular, it satisfies the Chapman-Kolmogorov eq. Let \( \alpha \) be a probability measure on \( S \). For \( 0 < t_1 < t_2 < \cdots < t_n \), define the finite dimension distribution of \( X_0, X_{t_1}, \cdots, X_{t_n} \) by

\[
P(X_0 \in B_0, X_{t_1} \in B_1, \cdots, X_{t_n} \in B_n) = \int_{B_0} \int_{B_1} \cdots \int_{B_{n-1}} P(t_n-t_{n-1}, x_{n-1}, B_n) P(t_{n-1}-t_{n-2}, x_{n-2}, dx_{n-1}) \cdots P(t_1, x_0, dx_1) \alpha(dx_0)
\]

where \( B_0, B_1, \cdots, B_n \) are measurable subsets of the state space \( S \). Use the Daniell-Kolmogorov extension theorem to show there is a stochastic process with these finite dimensional distributions. Note that we proved in class that for a Markov process with transition function \( P \), the finite dimensional distributions are given by the above equation.

7. Let \( X_t \) be a Markov process with transition function \( P(t, x, B) \). We let \( S^\Delta = S \cup \{\Delta\} \) with the topology defined as we did in class. So if \( S \) is compact, \( \Delta \) is an isolated point and if \( S \) is not compact, \( S^\Delta \) is the one point compactification of \( S \). Let \( A \subset S \) be a Borel set. Let

\[
\tau_A = \inf\{t : X_t \in A\} \quad (6)
\]

Take an initial distribution \( \alpha \) such that \( \alpha(A) = 1 \). Define a new process \( Y_t \) by \( Y_t = X_t \) for \( t < \tau_A \) and \( Y_t = \Delta \) for \( t \geq \tau_A \). Show \( Y_t \) is a Markov process. It is usually described as the process \( X_t \) killed when it exits \( A \).

8. For a topological space \( X \), \( D_X[0, \infty) \) denotes the space of functions from \( [0, \infty) \) into \( X \) which are right continuous and have left hand limits at all \( t \). Let \( S \) be a locally compact Hausdorff space. Let \( D \) be a dense subset of \( C_0(S) \). Let \( x : [0, \infty) \to S \). Prove that \( x \in D_S[0, \infty) \) if and only if \( f(x) \in D_R[0, \infty) \) for all \( f \in D \).

9. Let \( S \) be a locally compact, Hausdorff space which is separable. Let \( D \subset C_0(S) \) be dense. Prove that \( D \) has a countable subset which is still dense in \( C_0(S) \).

10. Let \( X_t \) be a Markov process with transition function \( P(t, x, B) \). Let \( f : S \to \mathcal{R} \) be a bounded random variable. Prove \( t \to f(X_t) \) is right continuous from \( [0, \infty) \) into \( L^1(\Omega, P) \). (We needed this to apply the Doob regularity thm.)