The 2d self avoiding walk

Bridges, strips and hitting densities

Tom Kennedy

Department of Mathematics, University of Arizona

Supported by NSF grant DMS- 0758649

Outline

- Definitions of self-avoiding walk (SAW)
- Bridge decomposition
- SAW in half plane $\rightarrow$ SAW in strip
- Simulations of SAW in strip
- More general “bridges” or cut curves
- SLE partition functions and hitting densities
- Lattice effects

Joint work with Greg Lawler, Ben Dyhr, Michael Gilbert, Shane Passon.

The SAW - "grand canonical ensemble"

Let $\delta > 0$ (the lattice spacing). We work on the lattice $\delta\mathbb{Z}^2$.

Let D be a bounded domain and $z, w \in \partial D$.

Take all nearest neighbor walks in D from z to w which do not visit any site more than once.

Weight a walk ω by $\beta^{|\omega|}$.

The number of SAW of length N in the full plane grows like β_c^{-N}.

Take $\beta = \beta_c$.

Normalize to get a probability measure.

Try to let lattice spacing go to zero to get a probability measure on curves in D from z to w.

Could also consider unbounded domain.

The SAW - “canonical ensemble”

Let D be an unbounded domain and $z \in \partial D$.

Think of the upper half plane with $z = 0$.

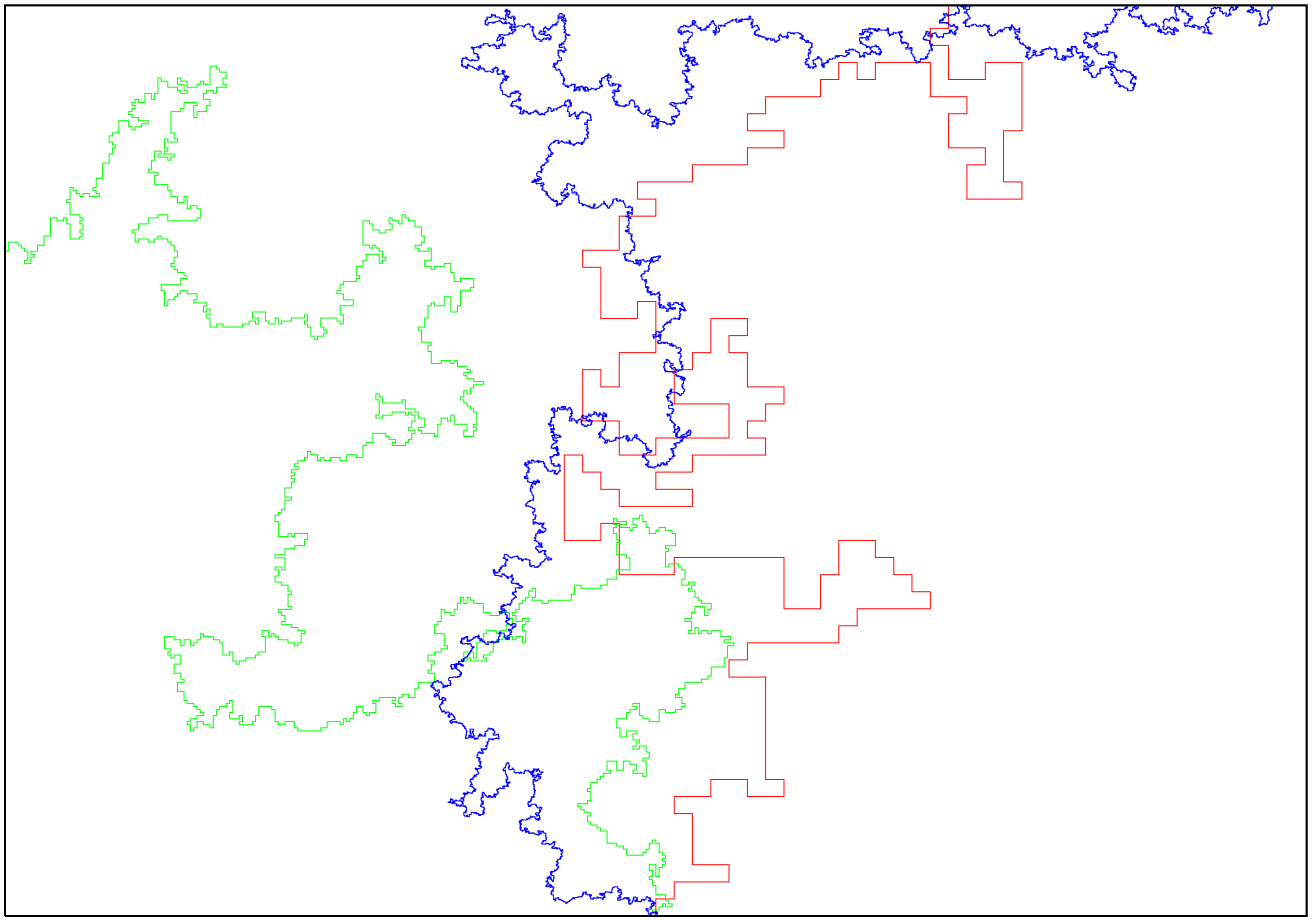
Take all nearest neighbor walks in D which start at z, have N steps and do not visit any site more than once.

Give them the uniform probability measure.

Try to let $N \to \infty$, then $\delta \to 0$.

If D is the uppper half plane, the $N \to \infty$ limit is known to exist.

SAW - scaling limit



$SAW = SLE_{8/3}$

Lawler, Schramm and Werner conjectured that the scaling limit of the SAW is $\mathrm{SLE}_{8/3}$.

Monte Carlo simulations of the SAW support this conjecture.

Past simulations have only been for canonical ensemble in the half plane or cut plane.

This talk: Simulations to test this conjecture for grand canonical ensemble in a strip.

The partition function point of view of SLE gives more predictions.

This talk: Simulations to test these predictions

What does SLE stand for ?

$SAW=SLE_{8/3}$

Lawler, Schramm and Werner conjectured that the scaling limit of the SAW is $SLE_{8/3}$.

Monte Carlo simulations of the SAW support this conjecture.

Past simulations have only been for canonical ensemble in the half plane or cut plane.

This talk: Simulations to test this conjecture for grand canonical ensemble in a strip.

The partition function point of view of SLE gives more predictions.

This talk: Simulations to test these predictions

What does SLE stand for ?

Schramm's paper: Stochastic Loewner Evolution

$SAW = SLE_{8/3}$

Lawler, Schramm and Werner conjectured that the scaling limit of the SAW is $SLE_{8/3}$.

Monte Carlo simulations of the SAW support this conjecture.

Past simulations have only been for canonical ensemble in the half plane or cut plane.

This talk: Simulations to test this conjecture for grand canonical ensemble in a strip.

The partition function point of view of SLE gives more predictions.

This talk: Simulations to test these predictions

What does SLE stand for ?

Schramm's paper: Stochastic Loewner Evolution

Lawler's book: Schramm-Loewner Evolution

$SAW=SLE_{8/3}$

Lawler, Schramm and Werner conjectured that the scaling limit of the SAW is $\mathrm{SLE}_{8/3}$.

Monte Carlo simulations of the SAW support this conjecture.

Past simulations have only been for canonical ensemble in the half plane or cut plane.

This talk: Simulations to test this conjecture for grand canonical ensemble in a strip.

The partition function point of view of SLE gives more predictions.

This talk: Simulations to test these predictions

What does SLE stand for ?

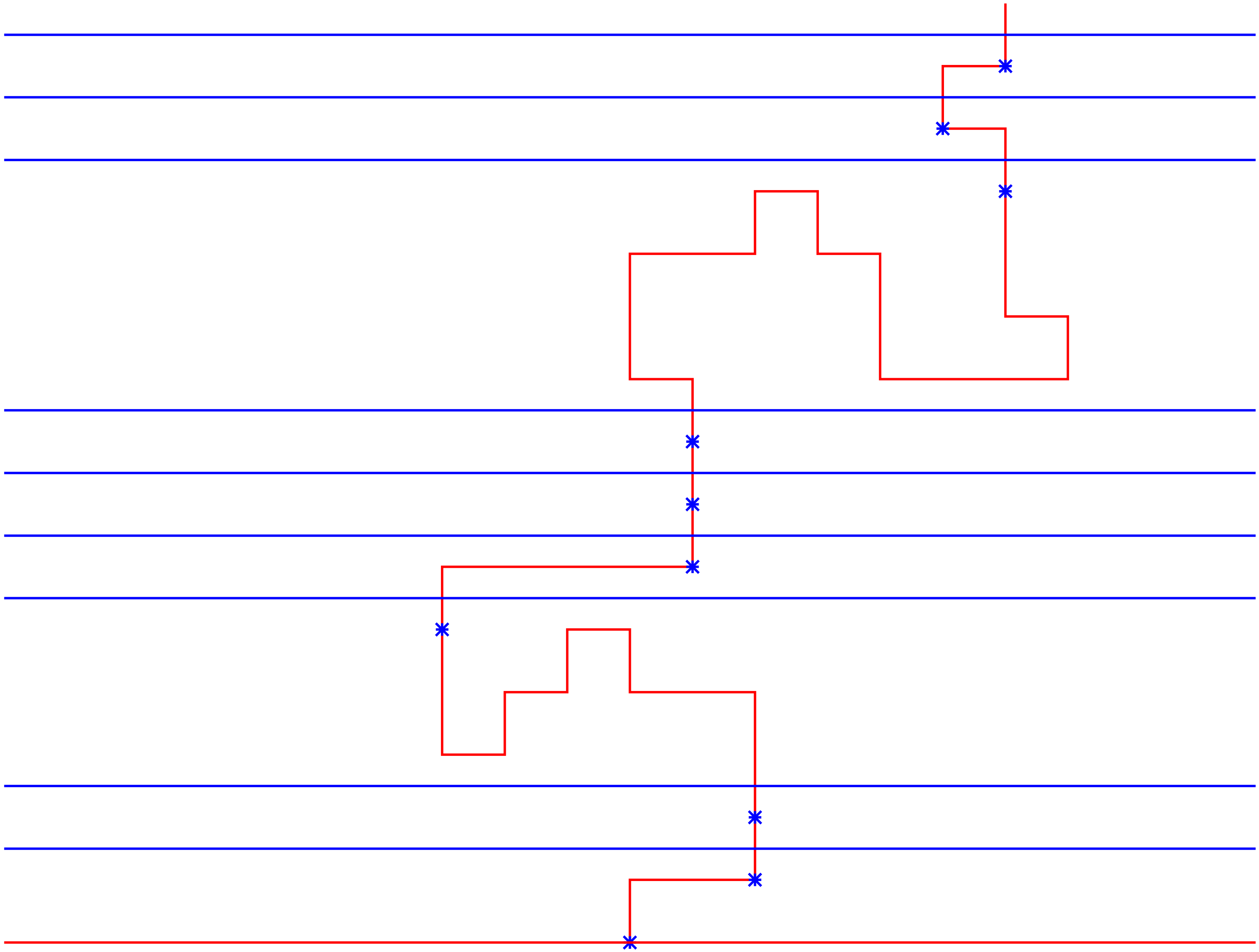
Schramm's paper: Stochastic Loewner Evolution

Lawler's book: Schramm-Loewner Evolution

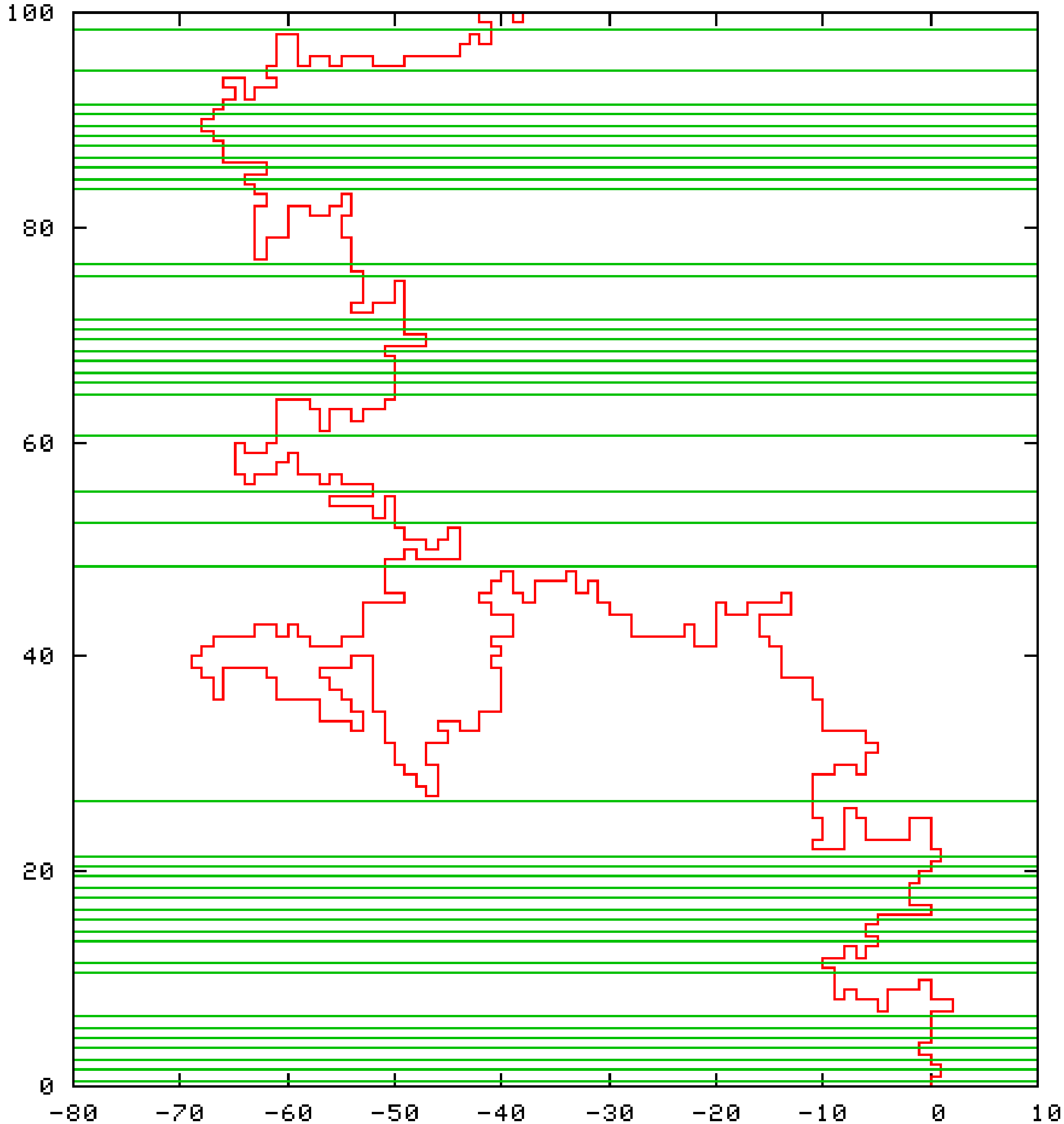
Google: Systemic lupus erythematosus

Bridges

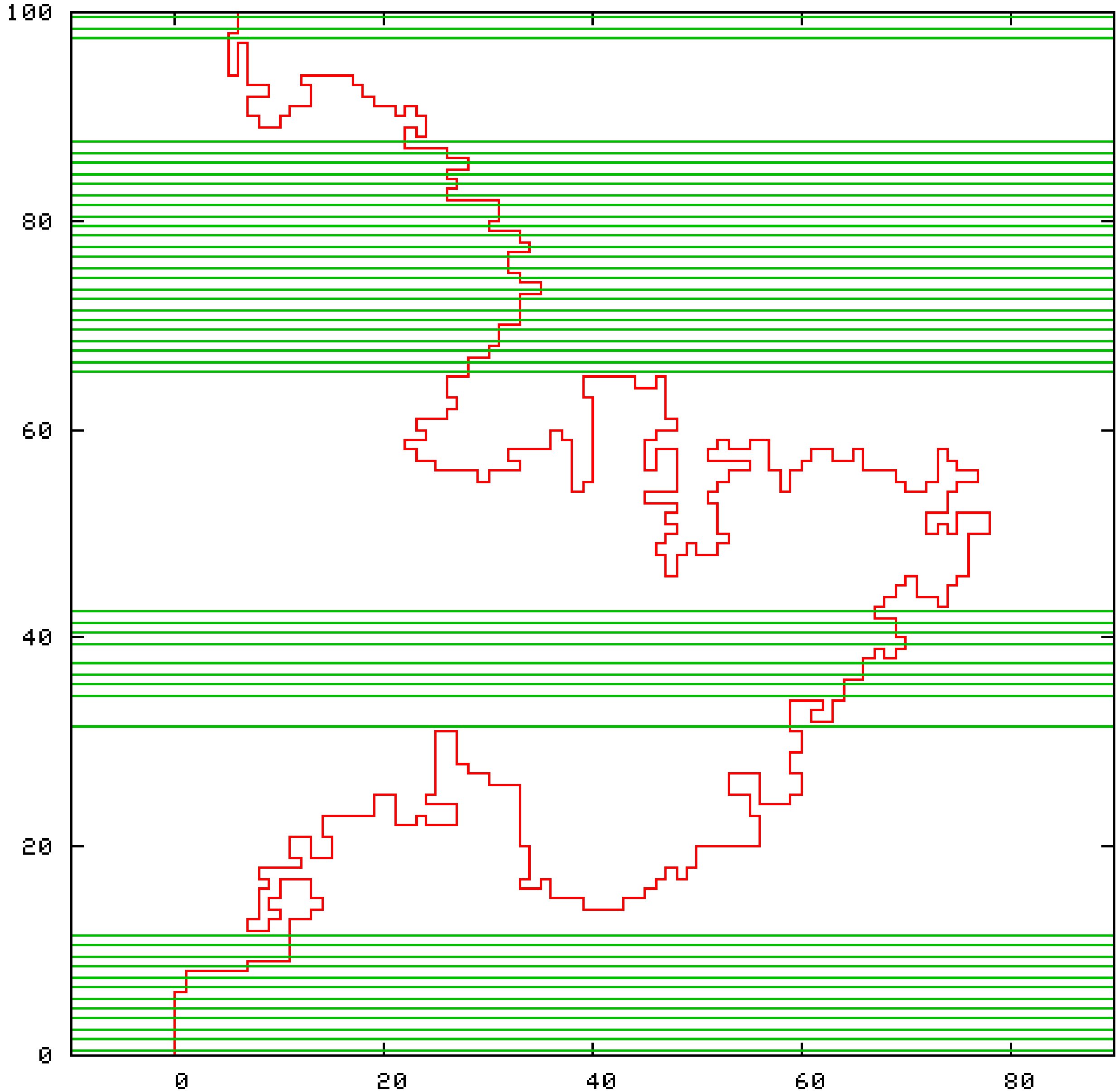
SAW in the half plane from 0



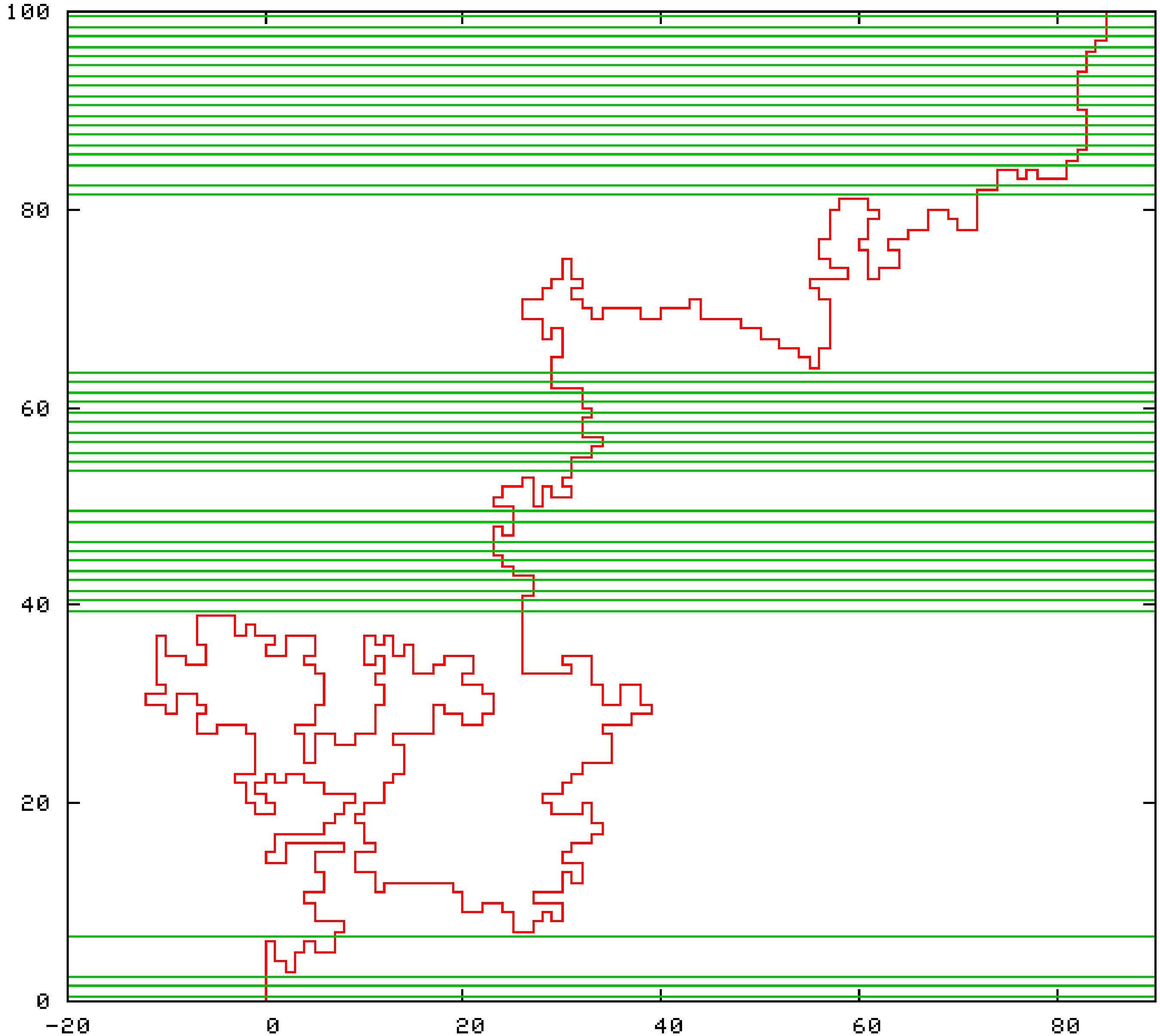
Bridges



Bridges



Bridges



Bridges

A bridge is irreducible if it is not the concatenation of 2 bridges.
Denote the set of irreducible bridges starting at 0 by $\mathcal{I}$.

Kesten

$$\sum_{\omega \in \mathcal{I}} \beta_c^{|\omega|} = 1$$

Madras and Slade used this to prove that the uniform measure on N-step bridges has a weak limit as $N \to \infty$.

Lawler, Schramm, Werner adapted this proof to show the canonical ensemble in $\mathbb{H}$ has a weak limit as $N \to \infty$.

This weak limit is given by the following construction.

Weight $\omega \in \mathcal{I}$ by $\beta_c^{|\omega|}$. This is a probability measure.
Let $\omega_n \in \mathcal{I}$ be i.i.d. sequence.
Concatenate them to get an infinite SAW.

Equivalence of ensembles

Continue to work in the upper half plane $\mathbb{H}$.

Consider the grand canonical ensemble for walks in $\mathbb{H}$ from 0 to ∞.

Normalization is infinite.

For $\beta < \beta_c$ the grand canonical ensemble makes sense.

Proposition: Letting $\beta \to \beta_c$ this grand canonical measure converges weakly to the probability measure from previous slide.

Conditioning: SAW in half plane $\rightarrow$ SAW in strip

How is half plane SAW related to SAW in strip?

Consider ordinary random walk conditioned to stay in upper half plane.

Stop it when it hits line of height y.

This is same as RW in the strip.

Do the same thing for SAW in $\mathbb{H}$ and you don't get SAW in strip.

Theorem Fix a positive integer y.

Condition P on the event that $y - 1/2$ is a cut line.

Only consider the walk up to height y

Then it has the distribution of the SAW in a strip

Conditioning: SAW in half plane $\rightarrow$ SAW in strip

Proof Let P be the probability measure on infinite SAW's in $\mathbb{H}$.
Let C_y be the event that ω has a cut at level y.
For $\omega_1, \ldots, \omega_j \in \mathcal{I}$, let $\mathcal{H}(\omega_1, \ldots, \omega_j)$ be the set of walks of the form $\omega_1 \otimes \cdots \otimes \omega_j \otimes \eta$ (concatenation).
where η is a walk in $\mathbb{H}$ starting at 0. NB ...

$$C_y = \cup_{j=1}^{y} \cup_{\omega_1, \ldots, \omega_j \in \mathcal{I} : h(\omega_1) + \cdots + h(\omega_j) = y} \mathcal{H}(\omega_1, \ldots, \omega_j)$$

We know

$$P(\mathcal{H}(\omega_1, \ldots, \omega_j)) = \beta_c^{|\omega_1| + \cdots + |\omega_j|}$$

so the proposition follows.

Simulations of SAW in strip

Fast algorithm for SAW in half plane - pivot algorithm - Clisby

No such fast algorithm for SAW in strip

Use the proposition to simulate SAW in strip

We condition on having a cut at height y

but not on the horizontal position

So simulation is not of chordal SLE, but rather an integral over x of chordal SLE's from 0 to $x + iy$ w.r.t. a hitting density $\rho(x)$.

How do you compute $\rho(x)$?

SLE as a probability measure is of no help.

SLE partition functions and hitting densities

Partition Functions, Loop Measure, and Versions of SLE
Lawler, Journal of Statistical Physics, **134**, 813-837 (2009)

Return to normalization for grand canonical SAW.
Call it $N(D, z, w, \delta)$

As $\delta \to 0$, it is believed to go to zero as δ^{2b}.

And $\lim_{\delta \to 0} N(D, z, w, \delta)\delta^{-2b}$ should exist.

Call it $C(D, z, w)$. This should be related to SLE partition functions which we denote as $Z(D, z, w)$.

SLE partition functions are conformally covariant:

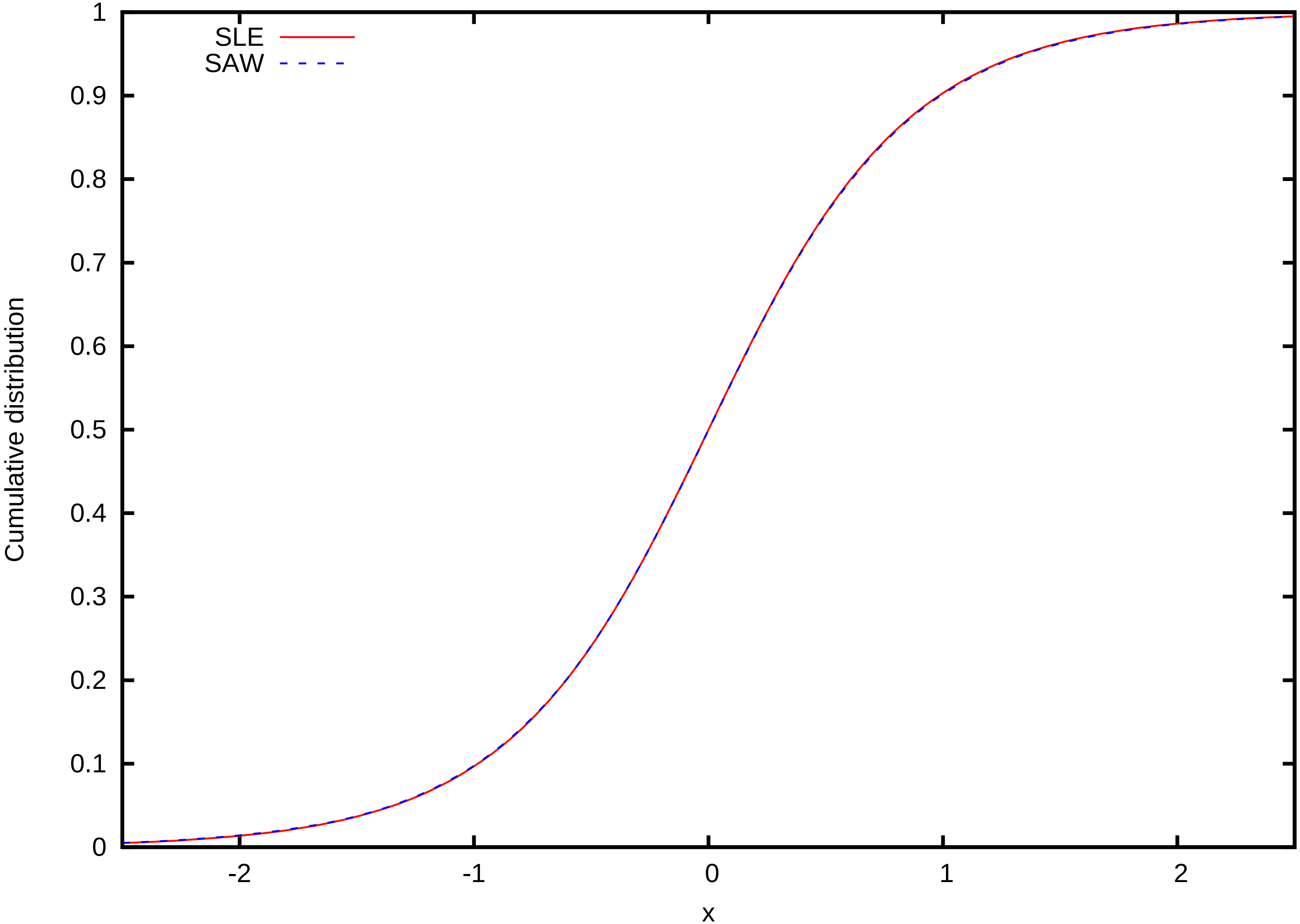
$$Z(D; z, w) = |f'(z)|^b |f'(w)|^b Z(f(D); f(z), f(w))$$

Is $C(D, z, w)$ proportional to $Z(D, z, w)$?

Conjecture:

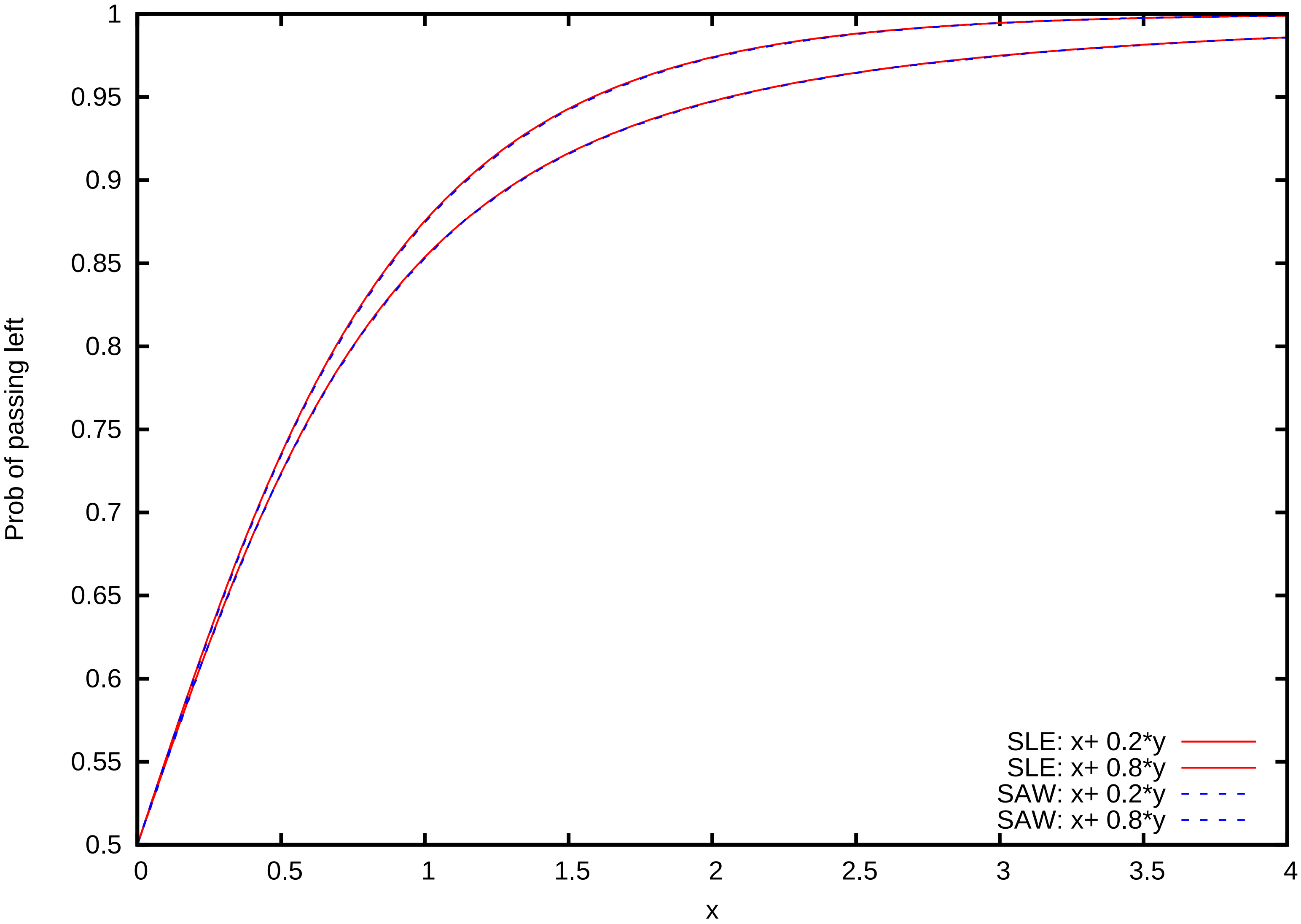
$$\rho(x) = c \left[\cosh\left(\frac{\pi x}{2y}\right) \right]^{-5/4}$$

Test of density conjecture



Test of passing left

Schramm computed the probability that chordal SLE passes left of a given point.



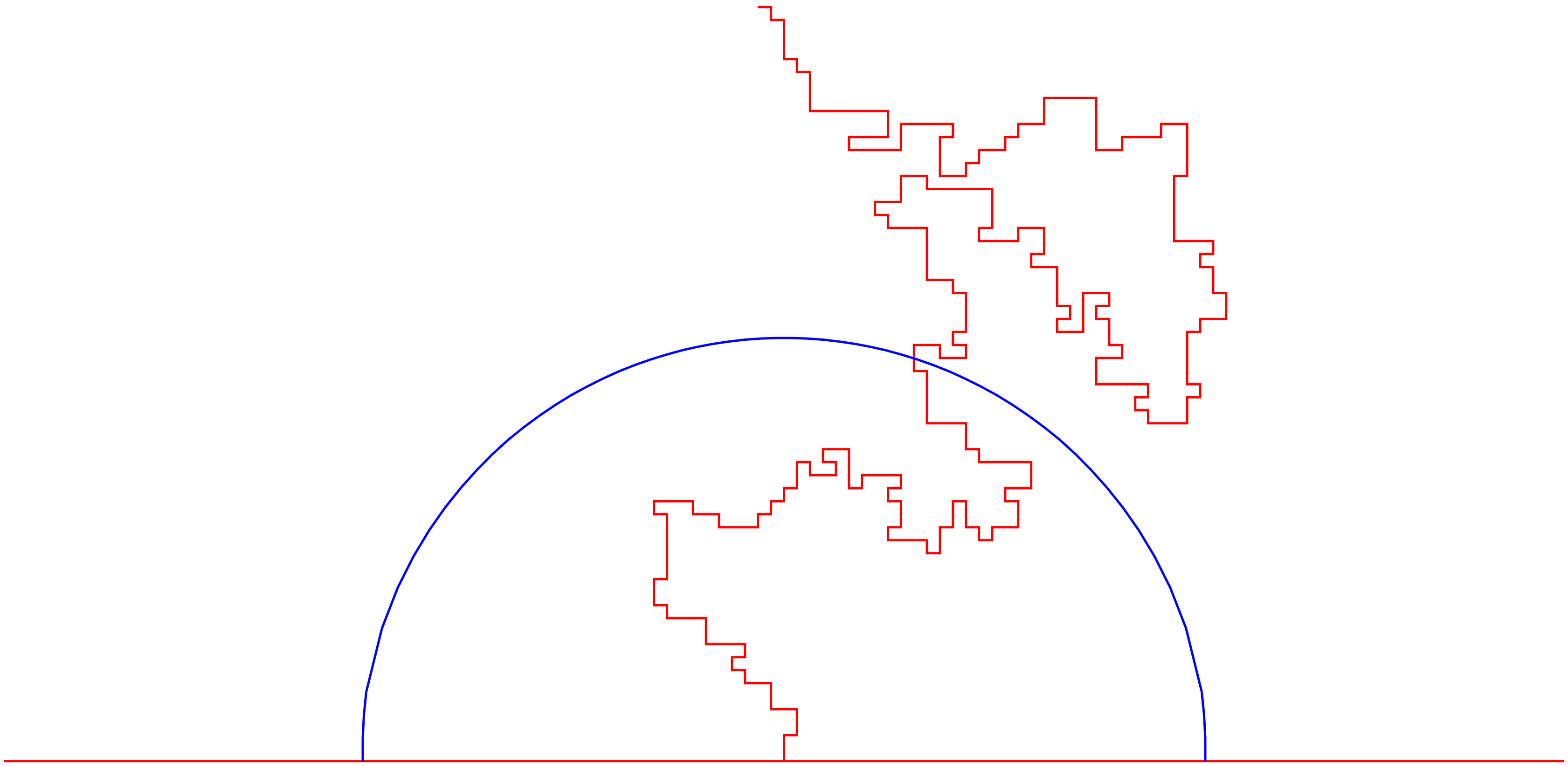
More general bridges or cut curves

Bridges: intersect horizontal line once

Generalize: horizontal line $\rightarrow$ another curve, e.g., semicircles in $\mathbb{H}$

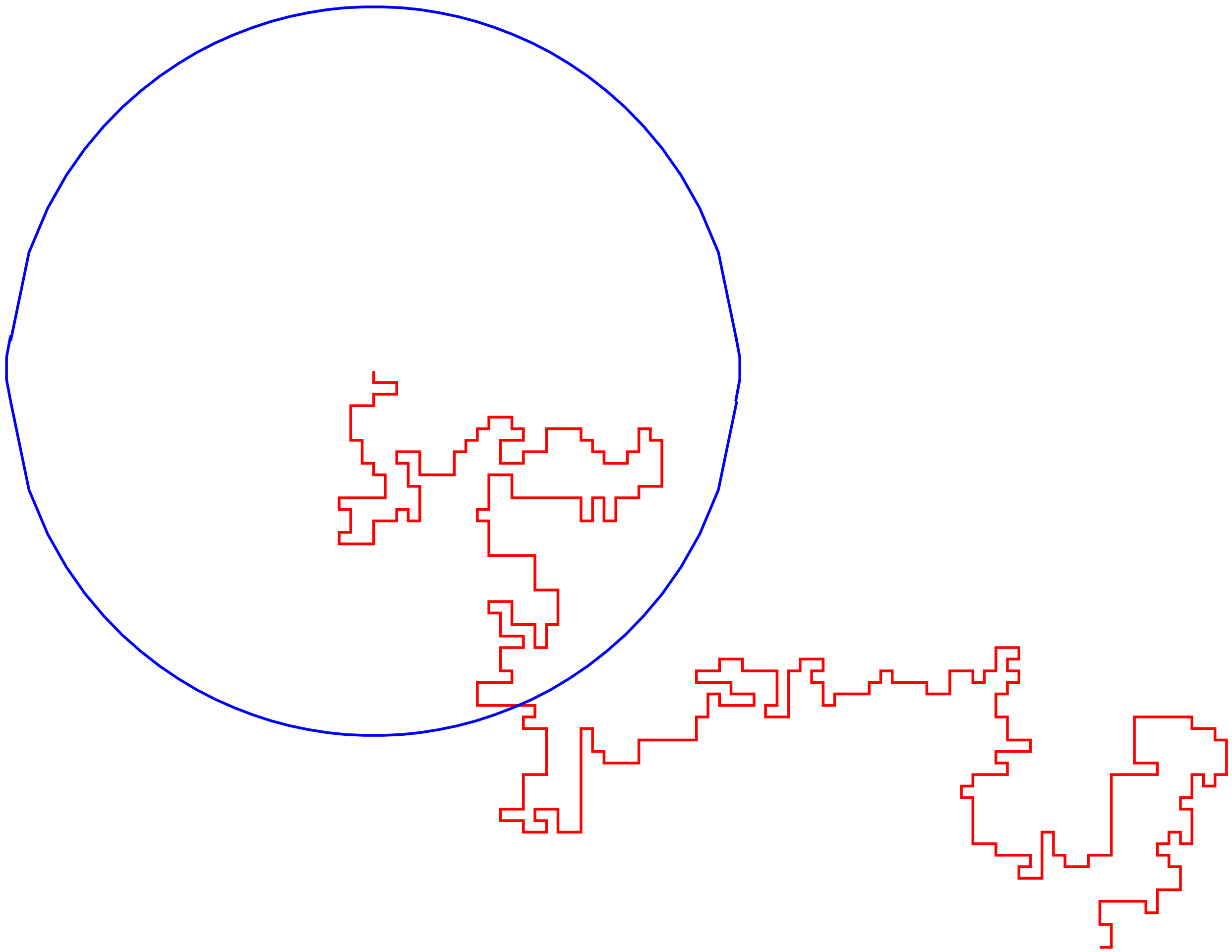
Look for SAW's that intersect the curve once

SLE partition functions predict $\rho(\theta) = [sin(\theta)]^{5/4}$.

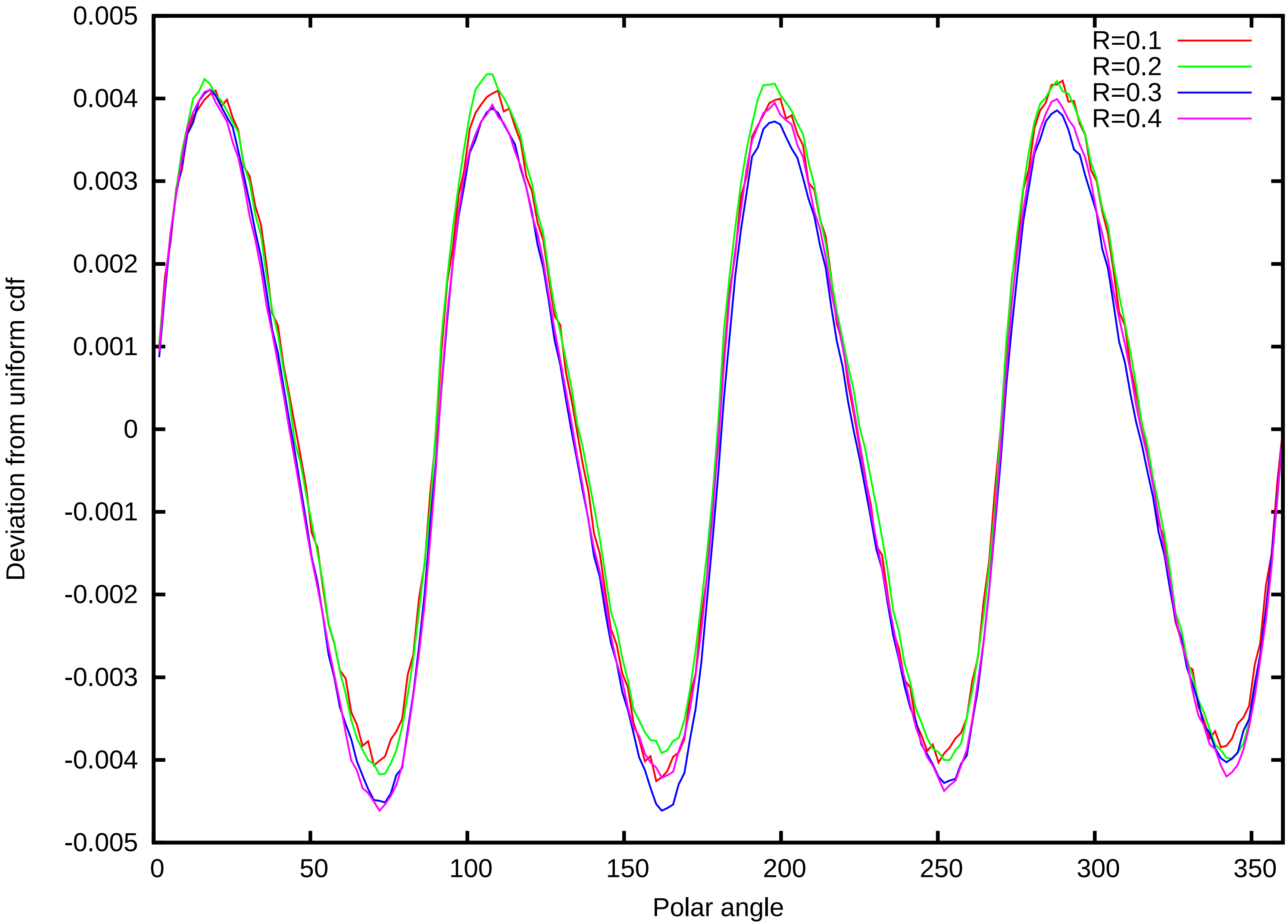


Lattice effects

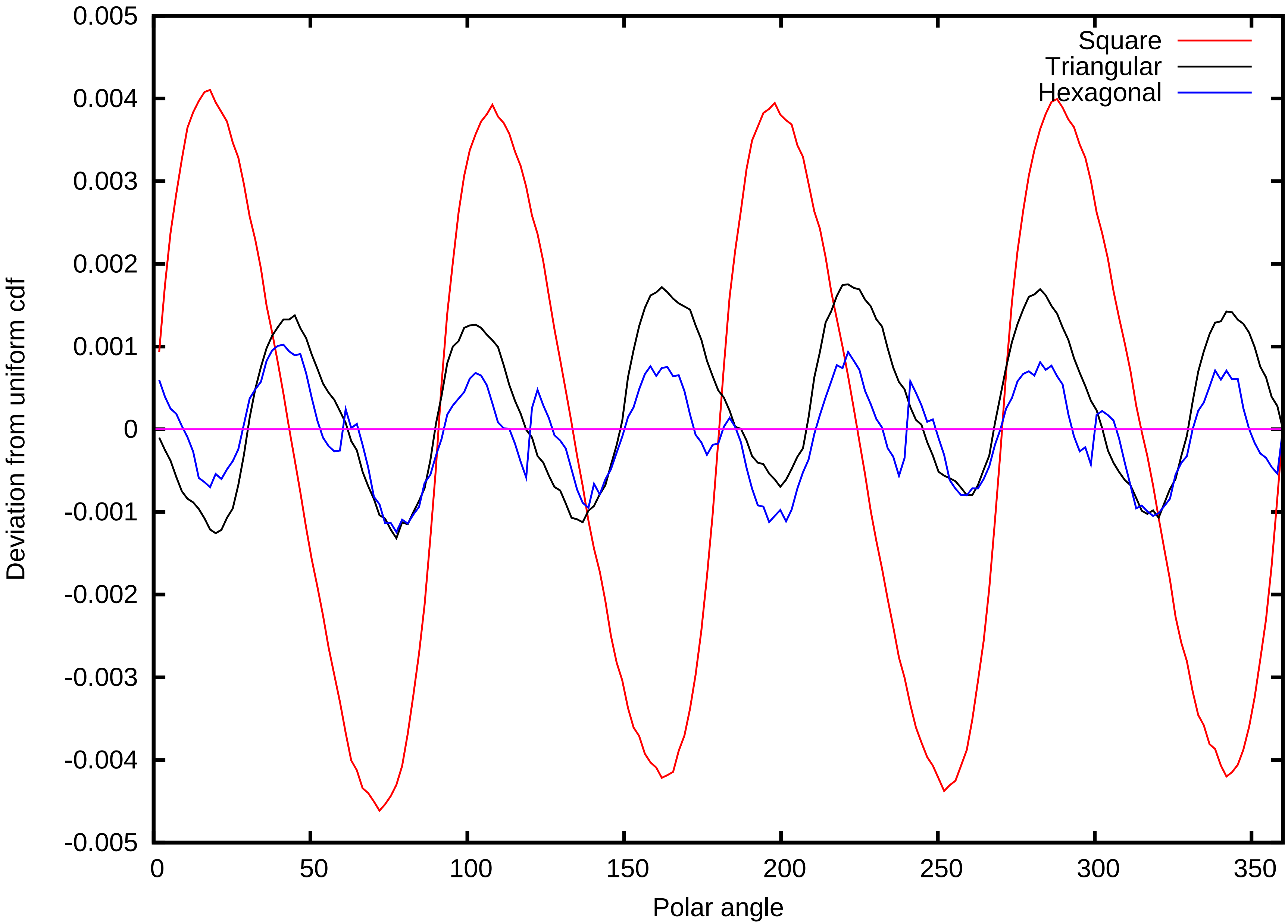
SAW in full plane
Condition to hit a circle exactly once.



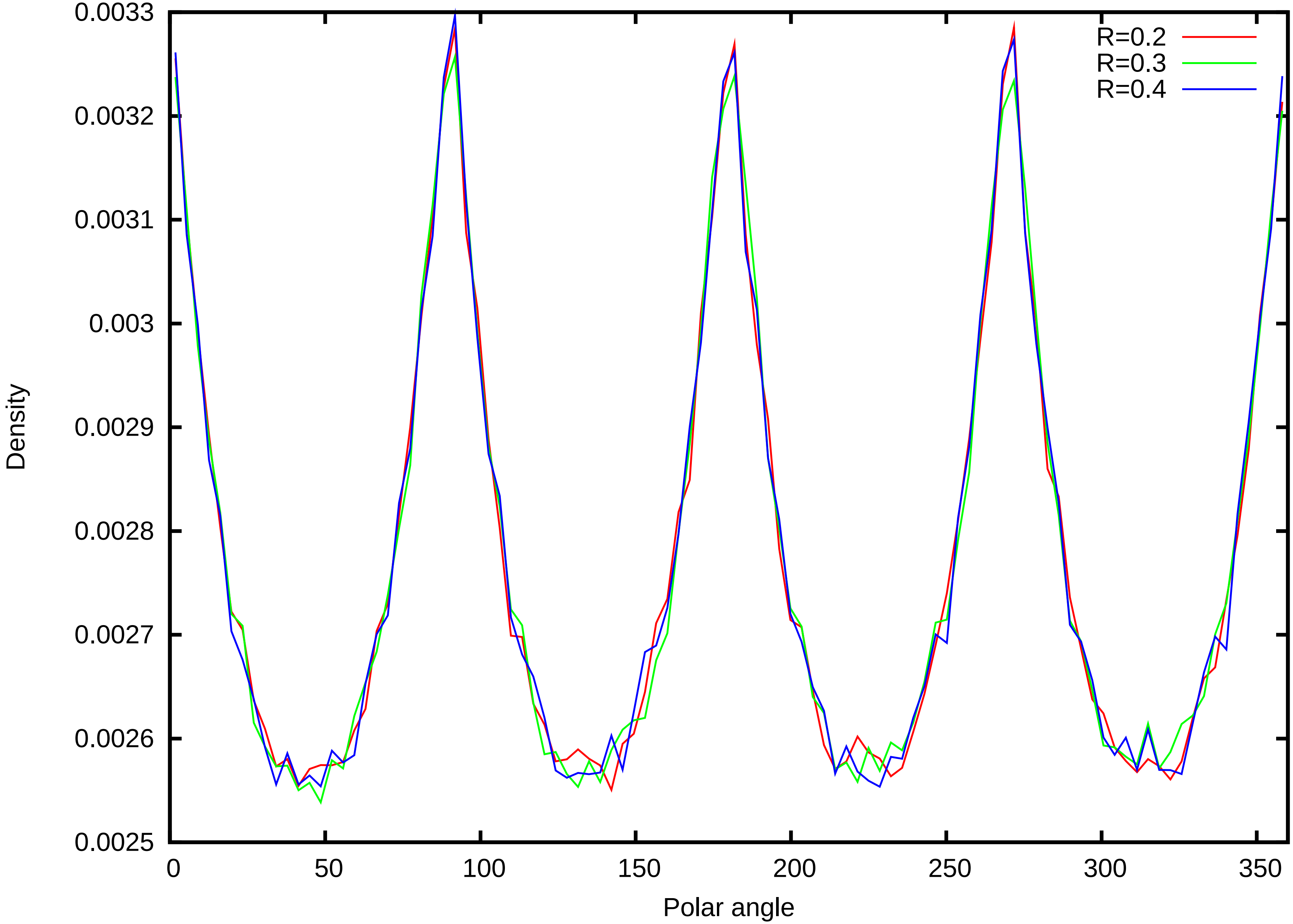
Lattice effects



Lattice effects

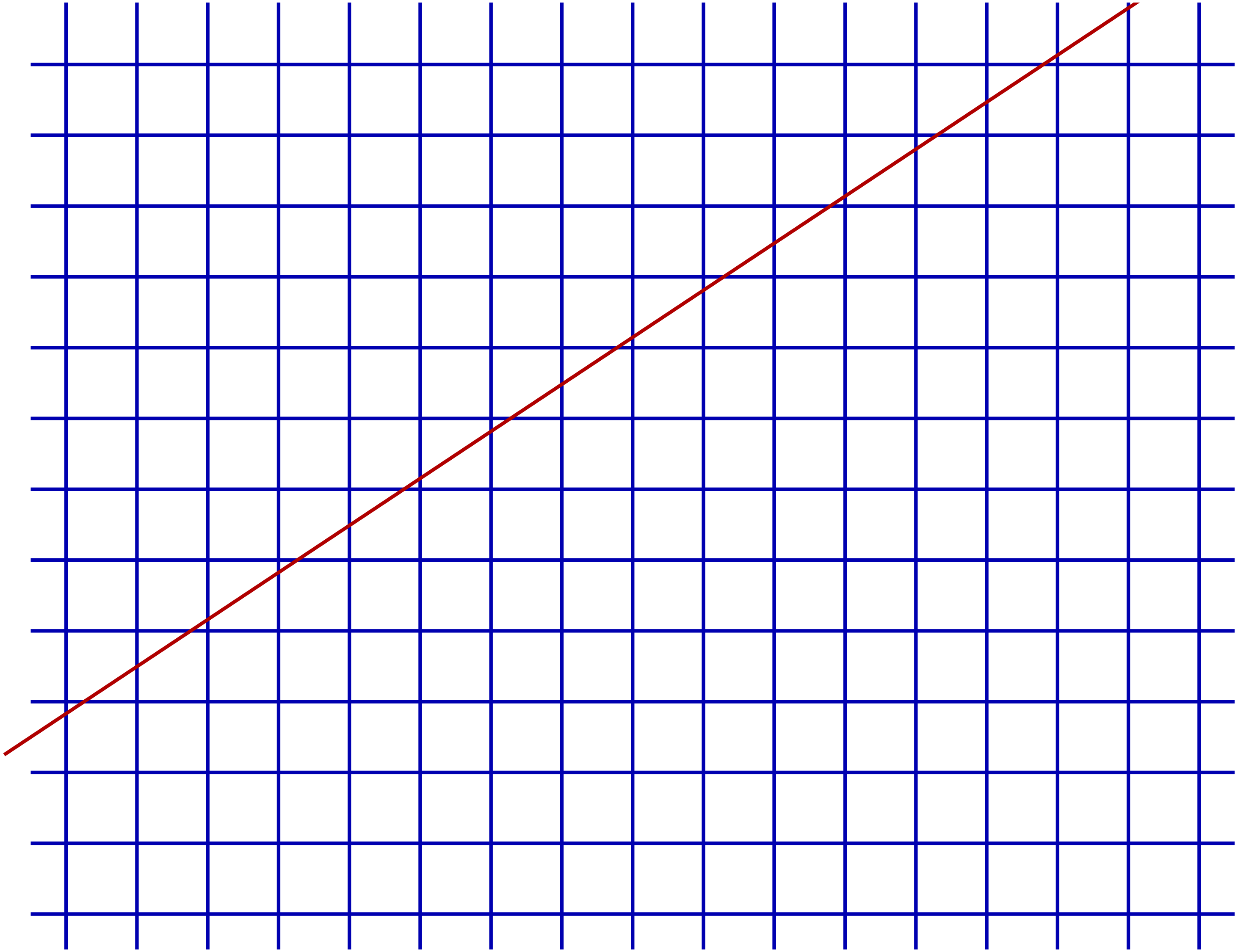


Lattice effects



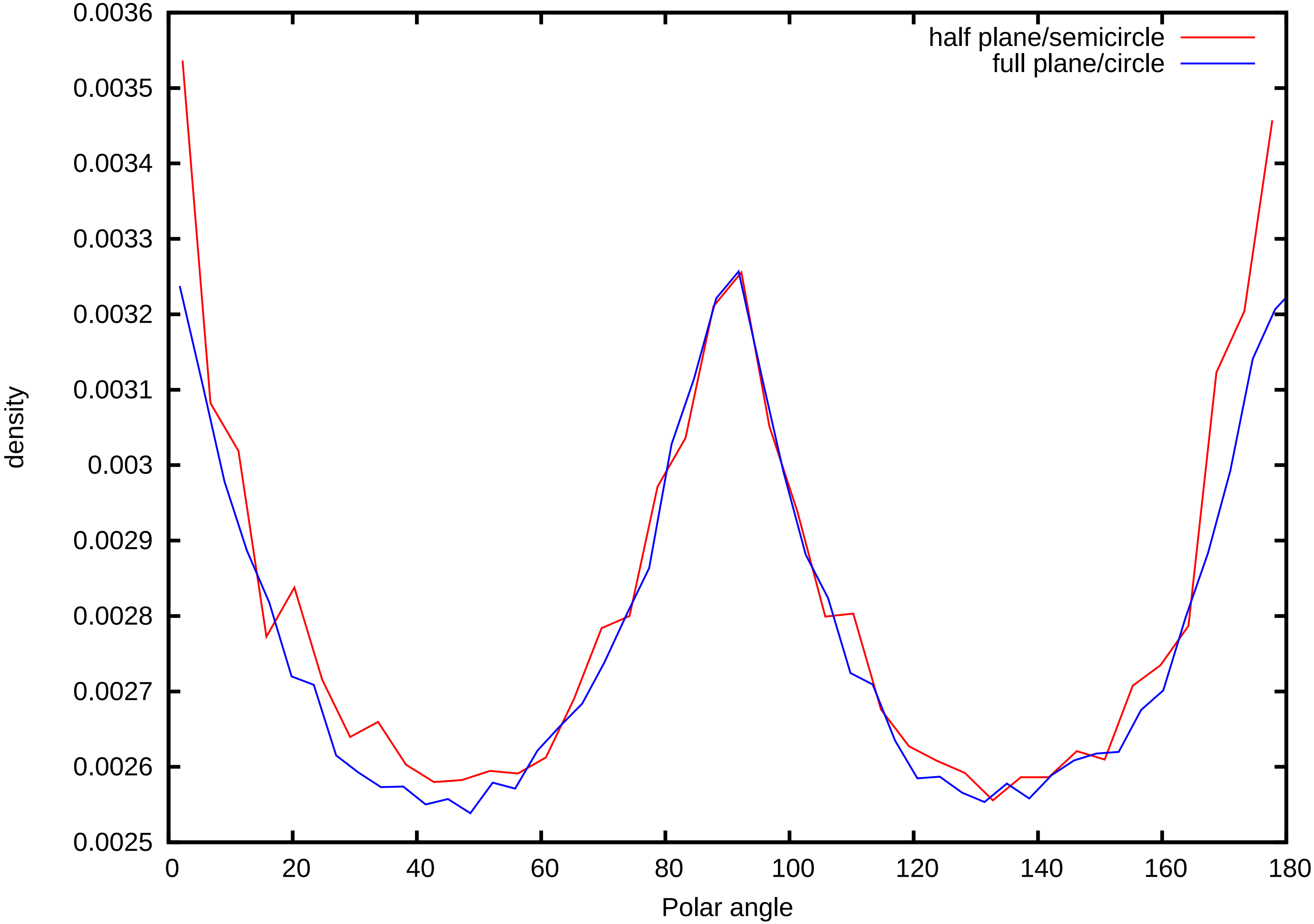
Lattice effects

Conjecture: Lattice effect is a factor $l(\theta)$ where θ is the angle between the boundary and the lattice directions.



Lattice effects

Comparison of $l(\theta)$ for SAW in full plane with cut on circle and SAW in half plane with cut on semicircle.



Conclusions, questions

- You can get SAW in a strip from SAW in $\mathbb{H}$ by conditioning
- Simluations of SAW in a strip agree with SLE
- SLE partition function prediction for hitting density for strip agree
- SLE partition function prediction for hitting densities for other geometries have lattice effects
- Is the conjecture on lattice effects correct?
- There are no rigorous results on the scaling limit of SAW
- Proofs of existence of $N \to \infty$ and $\beta \to \beta_c$ limits are only for $\mathbb{H}$