Monte Carlo comparisons of the self-avoiding walk and SLE

How should SLE be parameterized?

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The Self-Avoiding Walk

Take all N step, nearest neighbor walks in the upper half plane, starting at the origin which do not visit any site more than once.

Give them the uniform probability measure.

Let $N \to \infty$. Then let lattice spacing go to zero.

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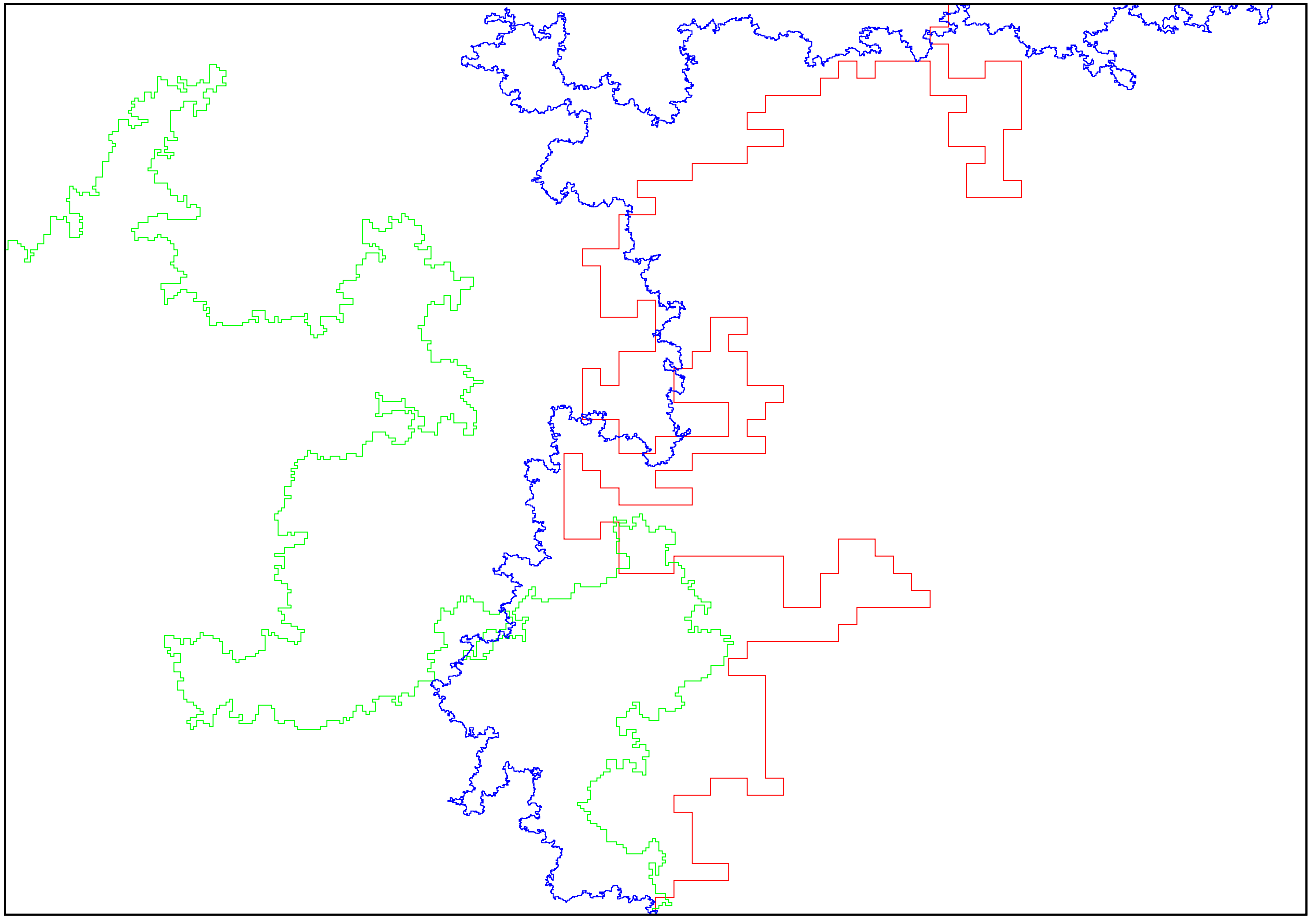

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(1) $$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z$$

B_t is a standard real Brownian motion, κ is a parameter.
z in upper half plane $\mathbb{H}$.

g_t is a random conformal map.

For $\kappa < 4$, domain of g_t is $\mathbb{H} \setminus \gamma[0,t]$.

Result: For each κ, a probability measure on curves in $\bar{\mathbb{H}}$ from 0 to ∞.

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References:

- *Conformally Invariant Processes in the Plane*, Greg Lawler, AMS.
- Wendelin Werner's 2002 St. Flour lectures
- Wouter Kager, Bernard Nienhius review article

$SAW = SLE_{8/3}$

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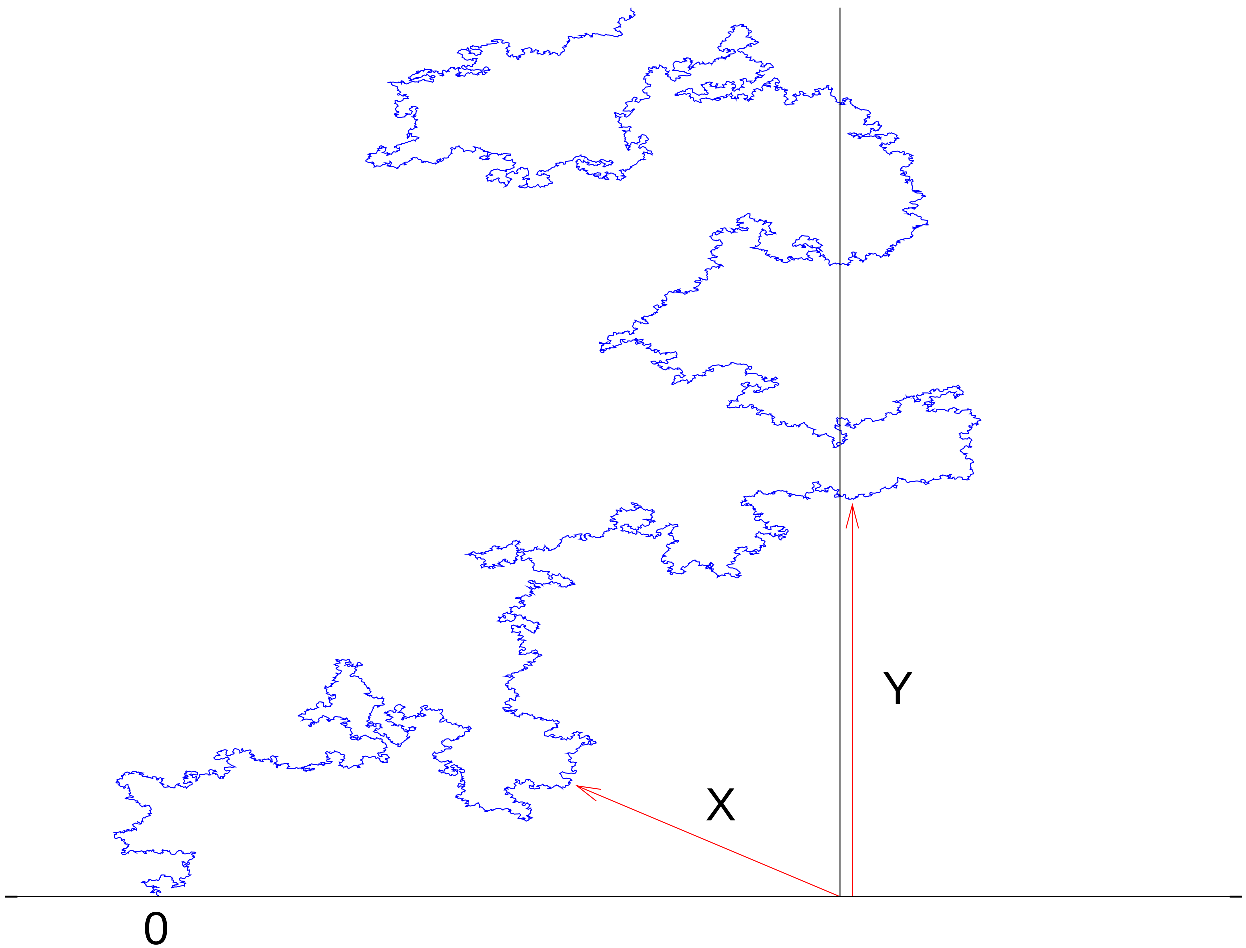
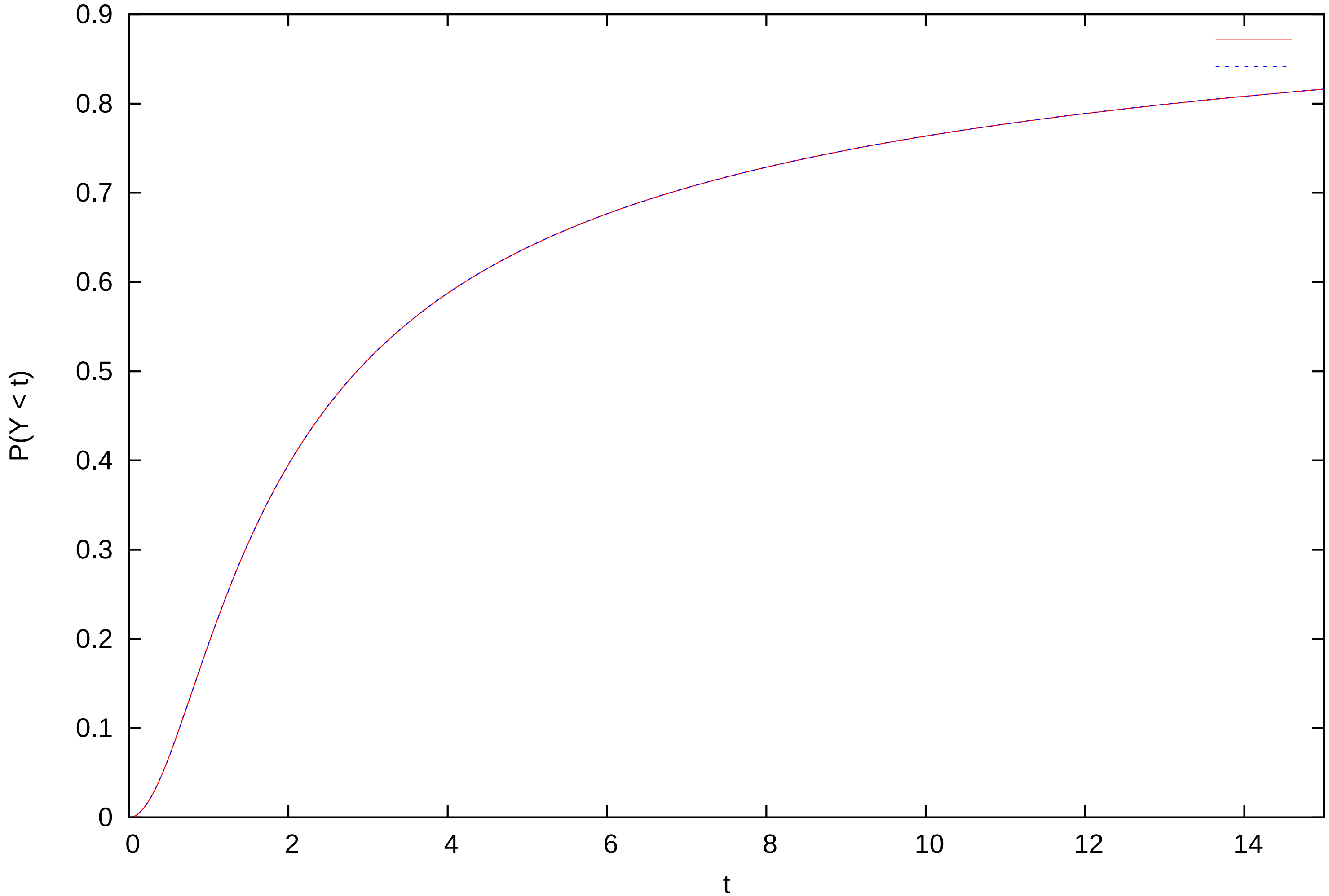


Distribution of Y

Every graph (but one) will be of the cumulative distribution function of a random variable Z, i.e., $P(Z \leq t)$ as a function of t.

All these graphs are the results of simulations. No theorems.



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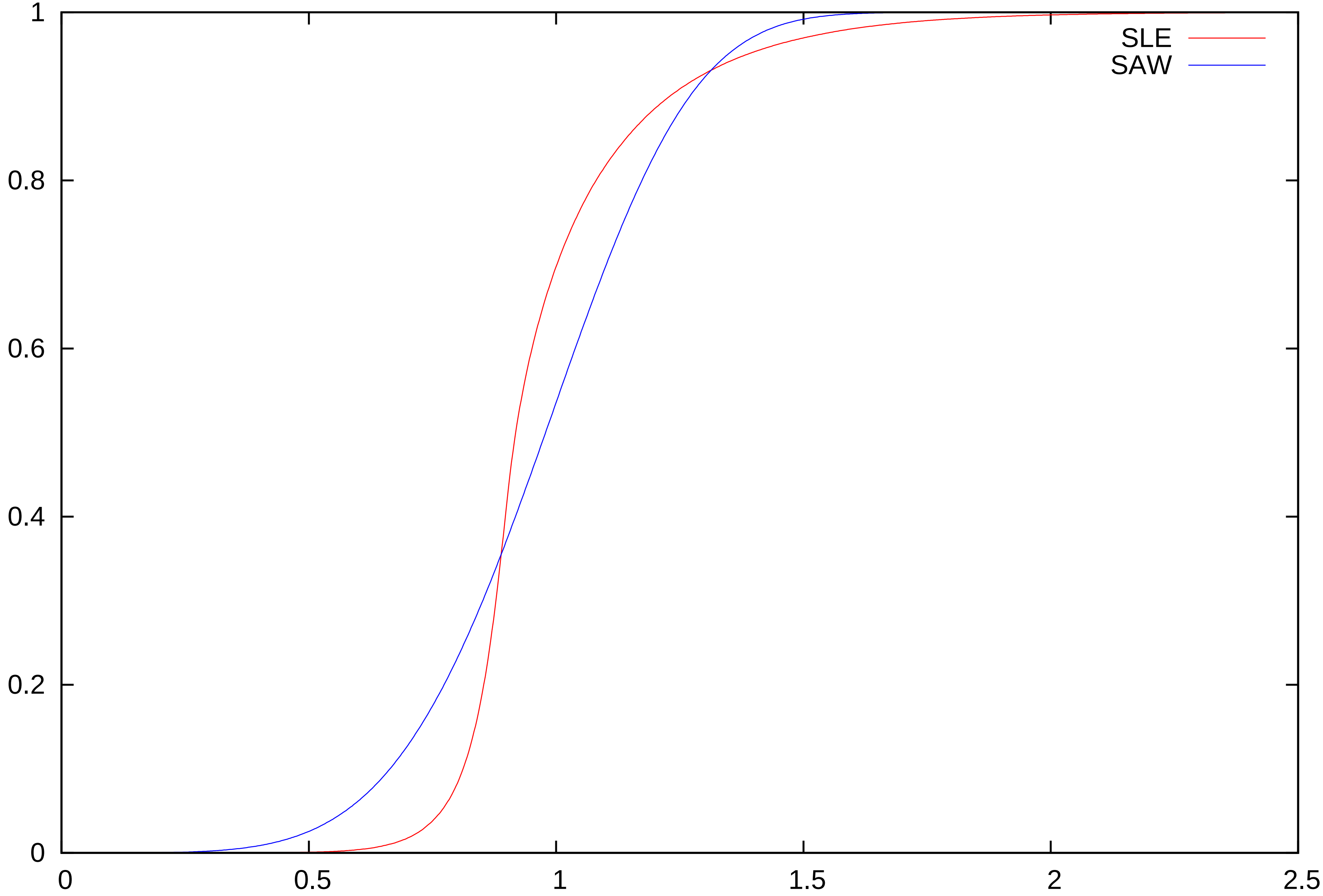
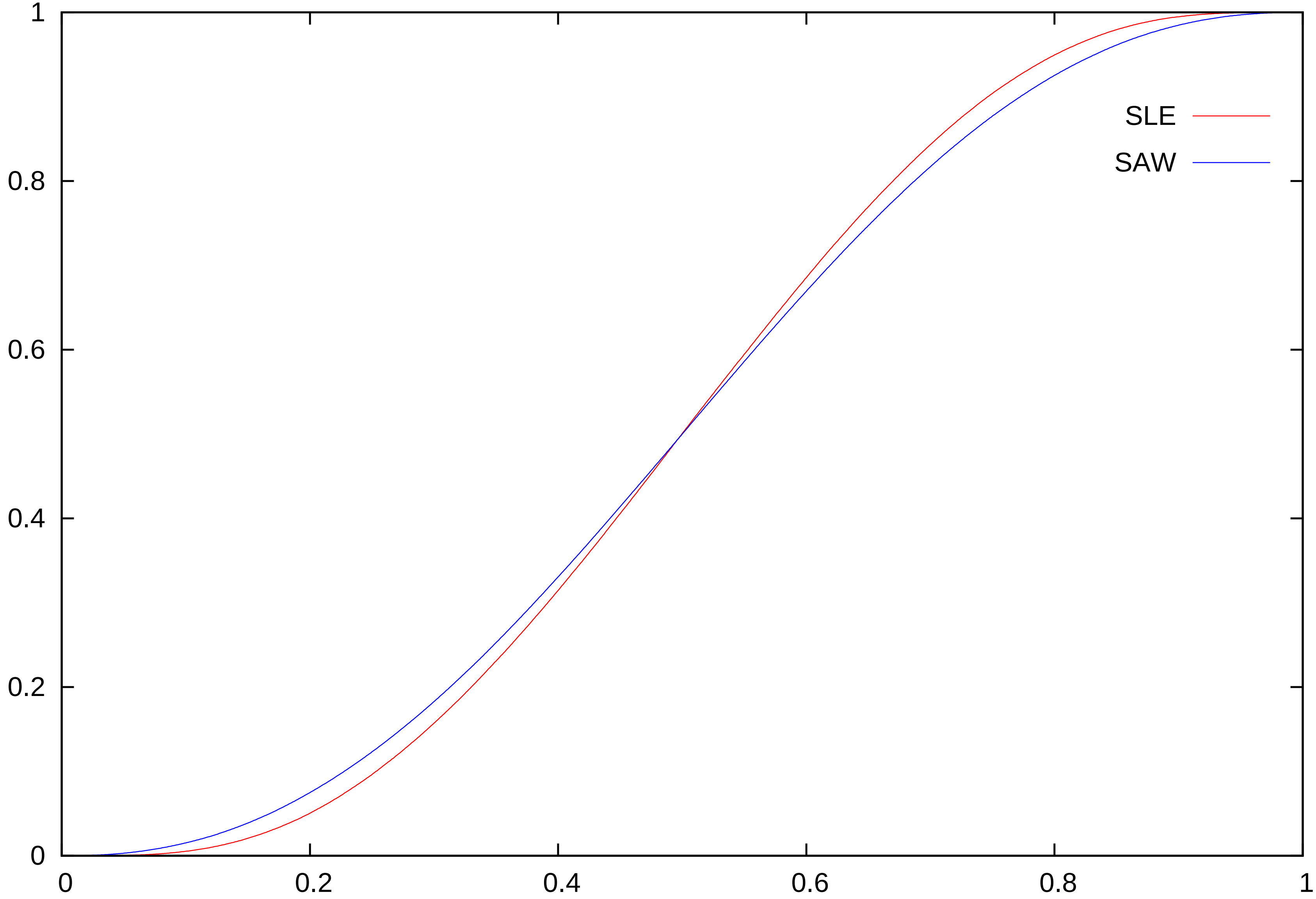
Compare $\gamma(1)$ and $\omega(1)$, rescaled so $|\gamma(1)|$ and $|\omega(1)|$ have mean one.

Distributions of distance from 0 *to* $\gamma(1), \omega(1)$



Distributions of polar angles of $\gamma(1), \omega(1)$



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- Easy question: Keep the capacity parameterization for SLE, reparameterize SAW.

Let C be the random time on the SAW where

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Compare $\omega(C)$ and $\gamma(1)$.

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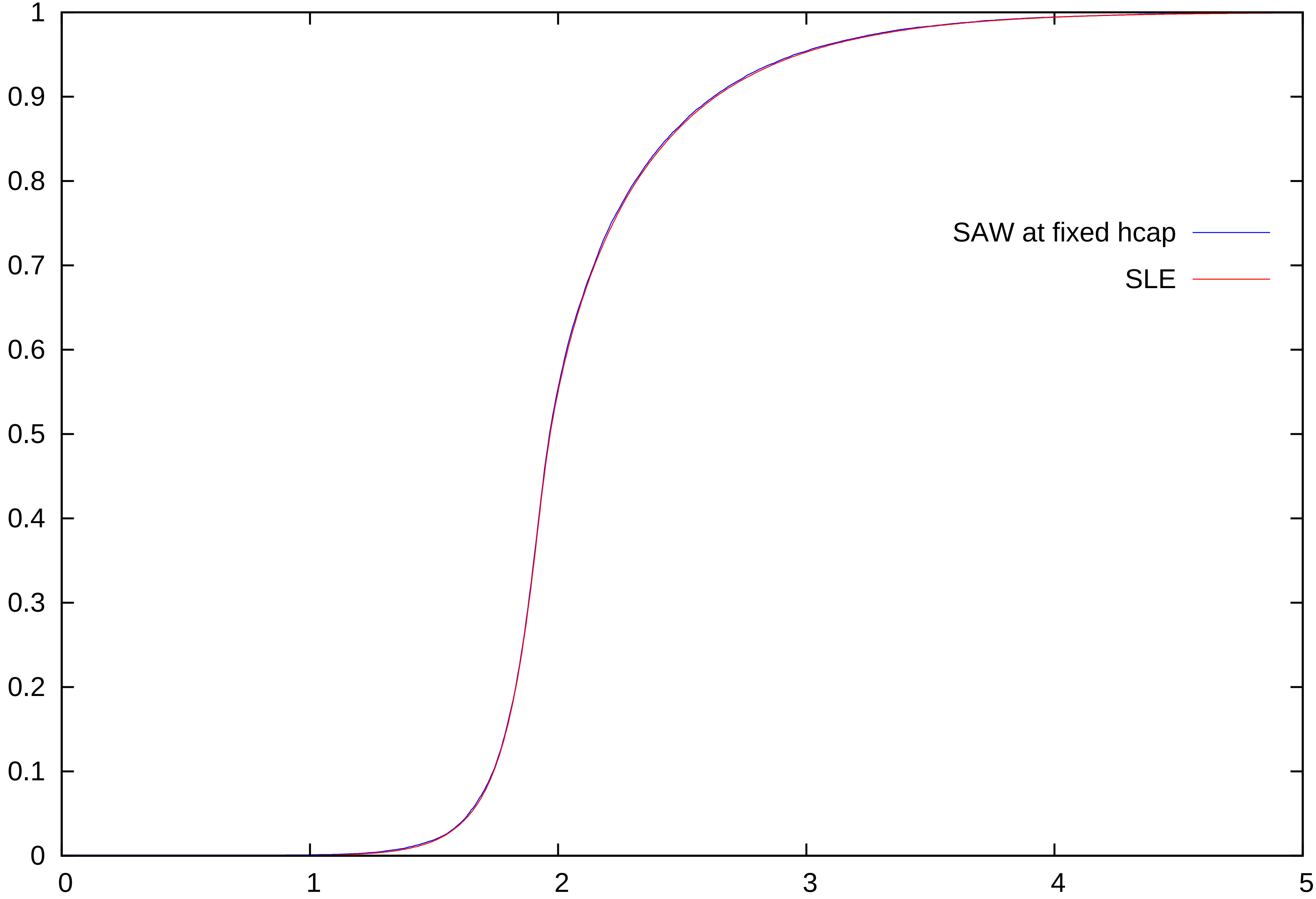
- Interesting question: Use natural parameterization for SAW, reparameterize SLE.

Distributions of distances for $\gamma(1), \omega(C)$



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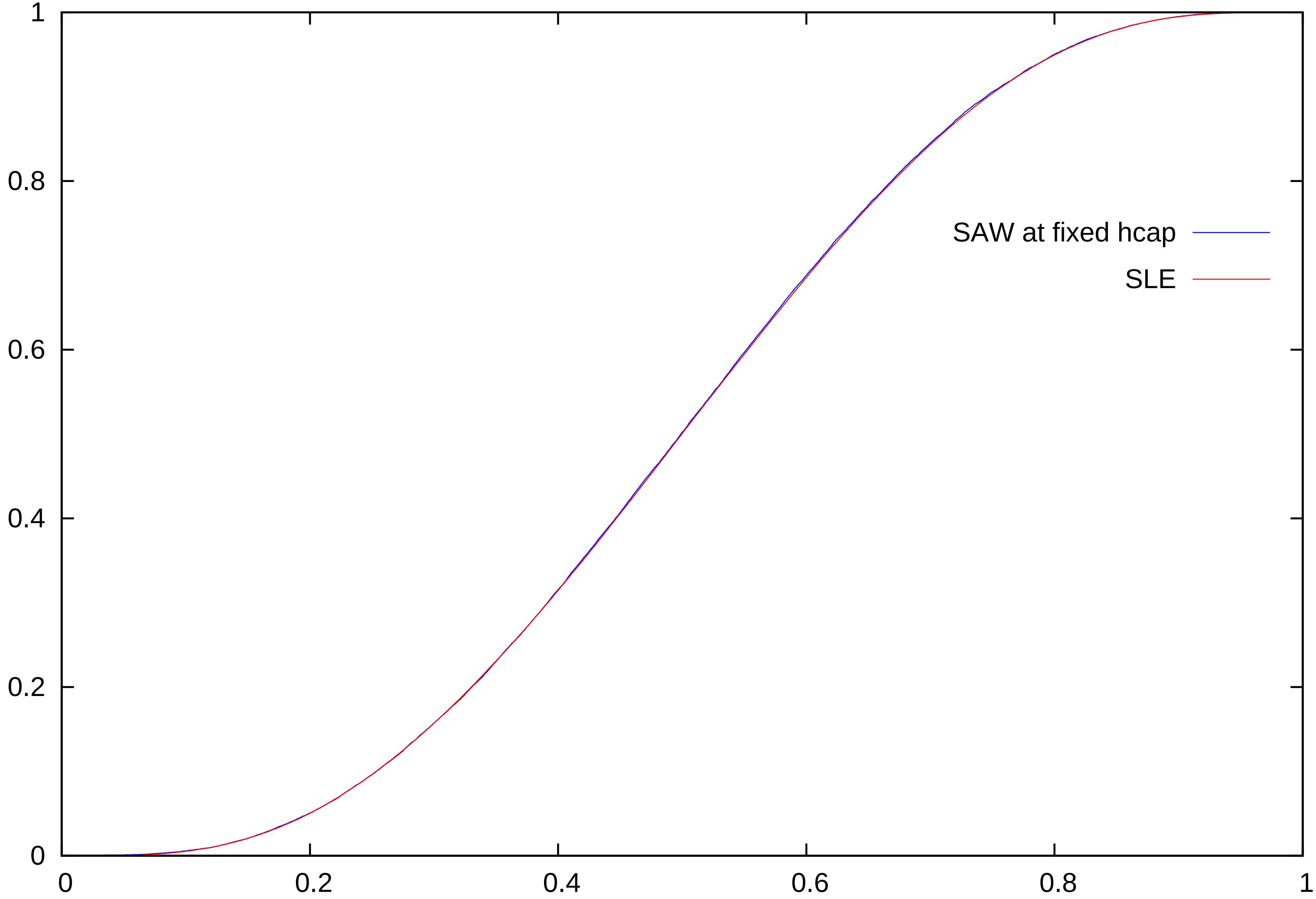

How to reparameterize SLE

Use pth variation with $p = 1/\nu$:
Let $0 = t_0^n < t_1^n < t_2^n \cdots < t_{k_n}^n = t$ be sequence of partitions of $[0, t]$.

$$fvar(\gamma[0,t]) = \lim_{n \to \infty} \sum_j |\gamma(t_j^n) - \gamma(t_{j-1}^n)|^{1/\nu}$$

With $\nu = 1/2$ this is the quadratic variation. For Brownian motion it is non-random and proportional to t.

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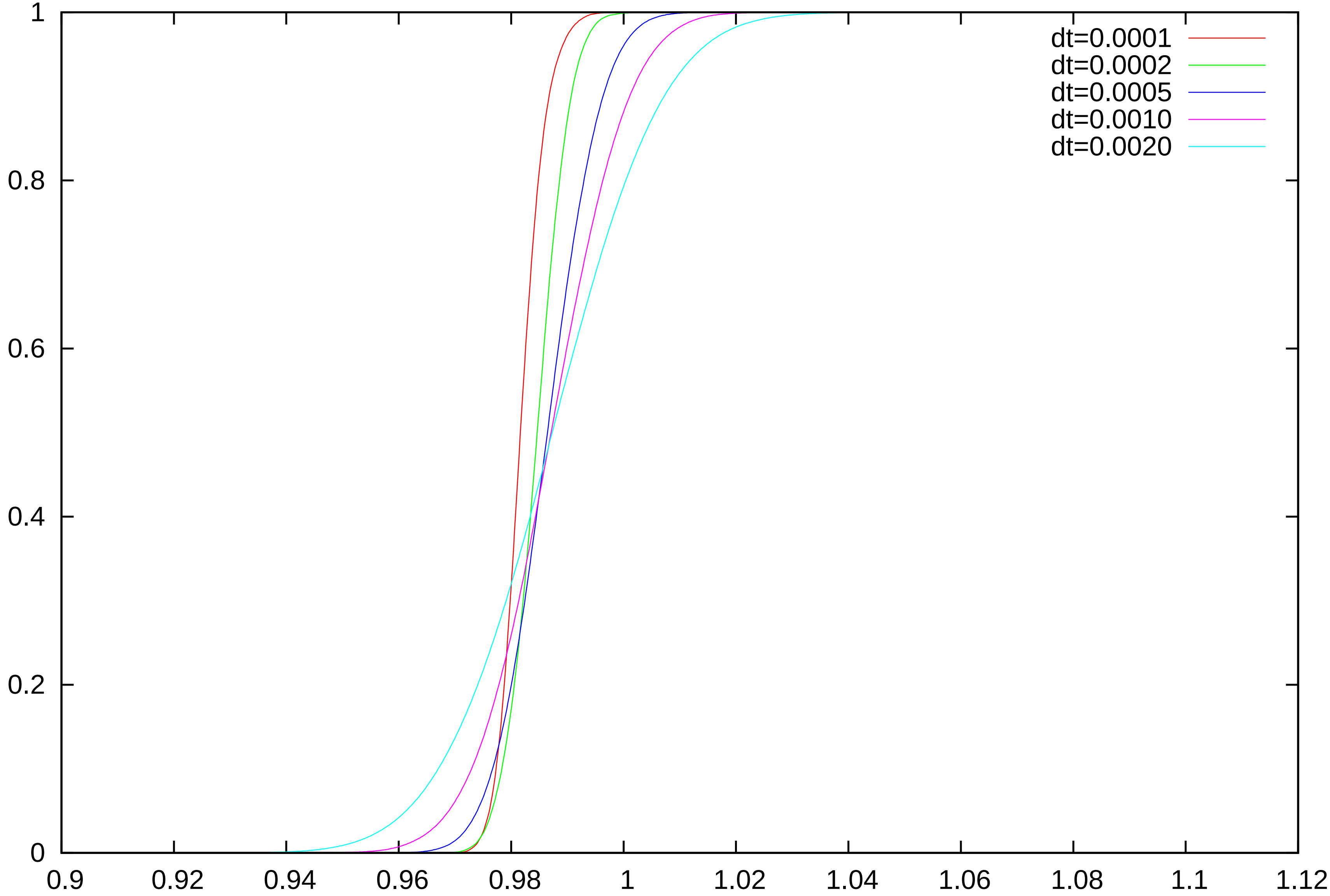
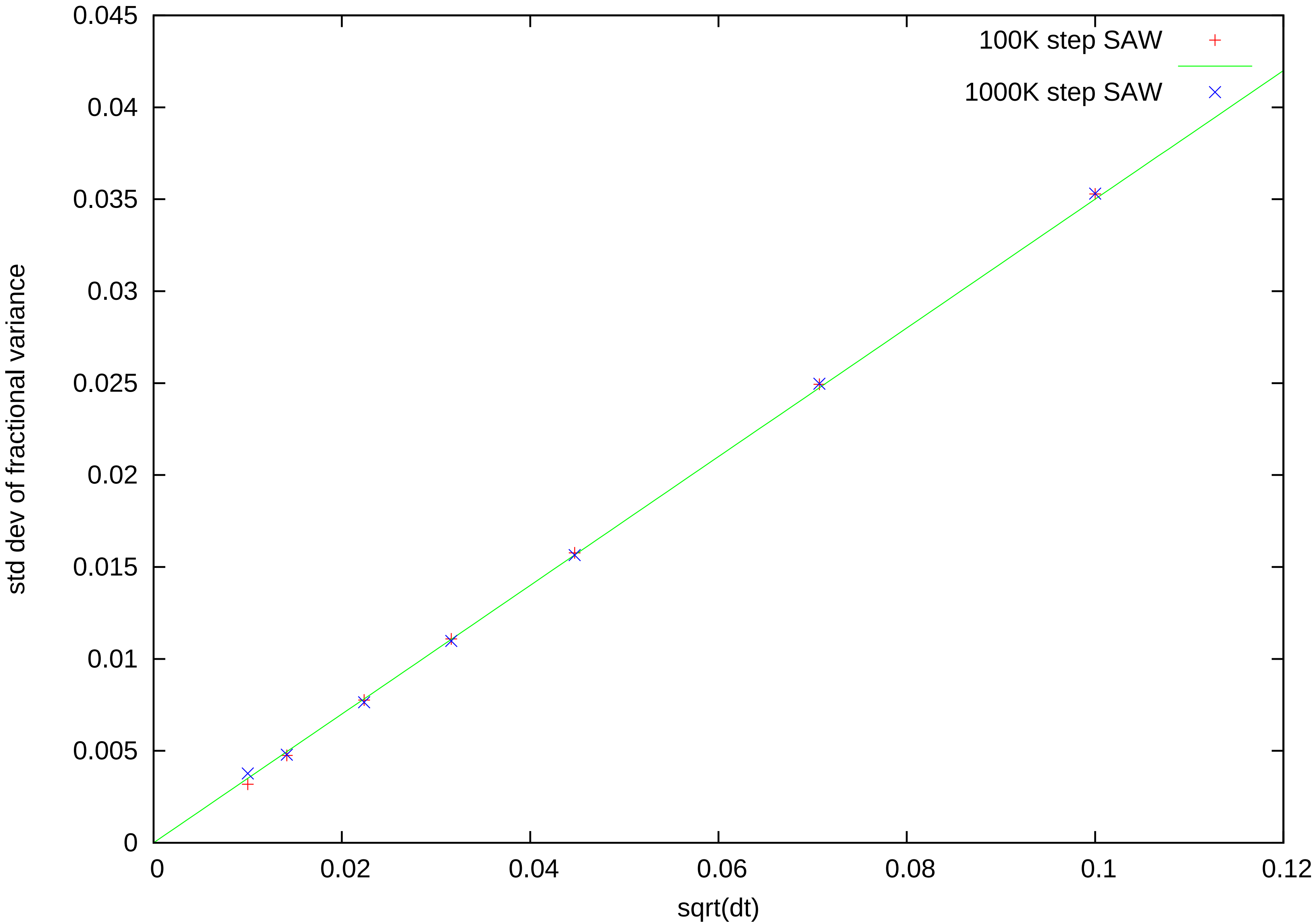
Key point: The $1/\nu$ variation of $\omega[0,t]$ is not random.

$1/\nu$ variation of SAW at fixed time



Standard deviation of frac. var. vs $\sqrt{dt}$



Parameterizing SLE by $1/\nu$ variation

Let T be the random time on the SLE where

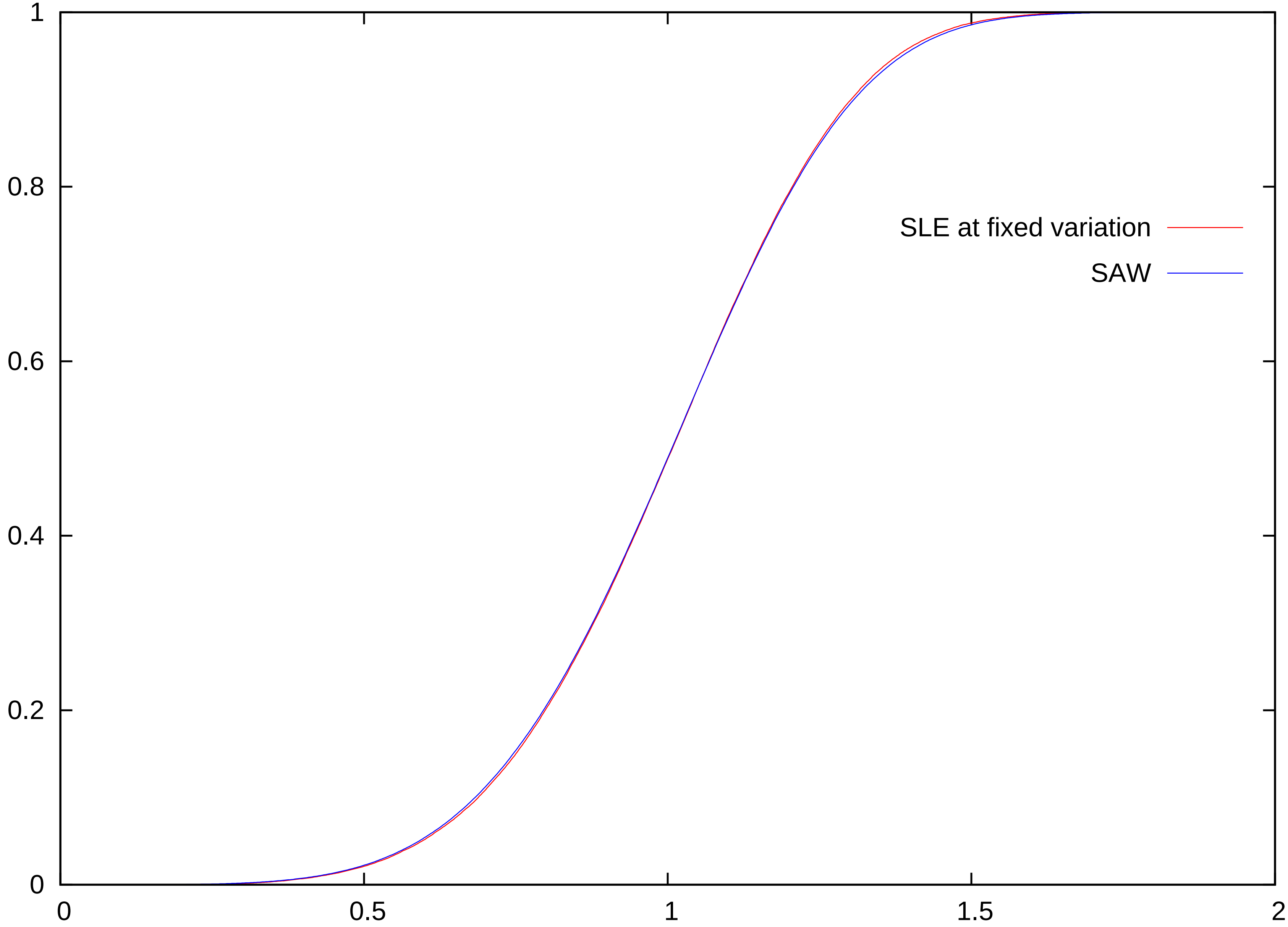
(5) $$fvar(\gamma[0,T]) = c$$

where $c = fvar(\omega[0,1])$.
Compare $\omega(1)$ and $\gamma(T)$.

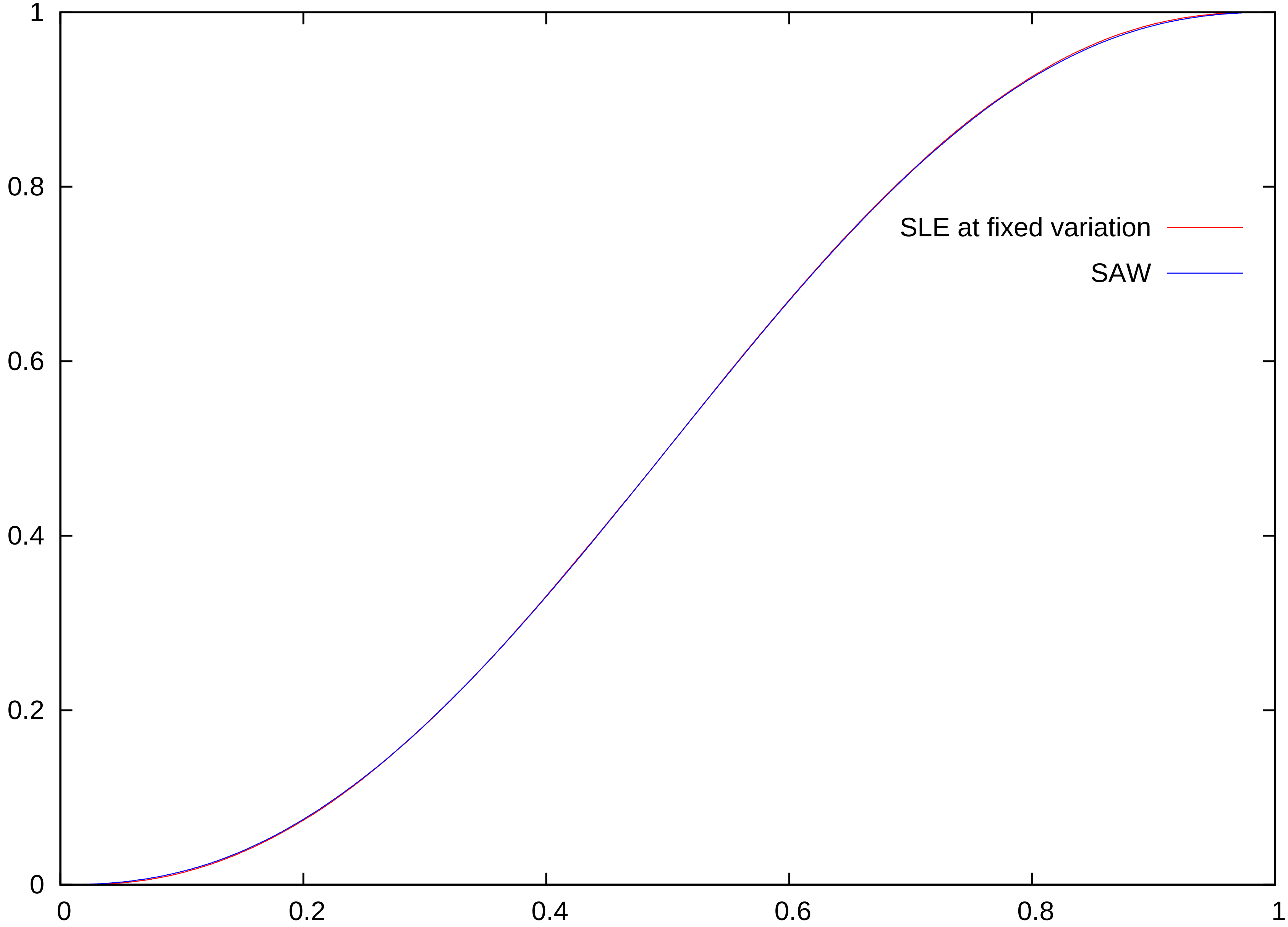
Cheat Rescale $\omega(1)$ and $\gamma(T)$ so that $|\omega(1)|$ and $|\gamma(T)|$ both have mean one.

This should not be necessary.

Distributions of distances for $\gamma(T), \omega(1)$



Distributions of angle for $\gamma(T), \omega(1)$



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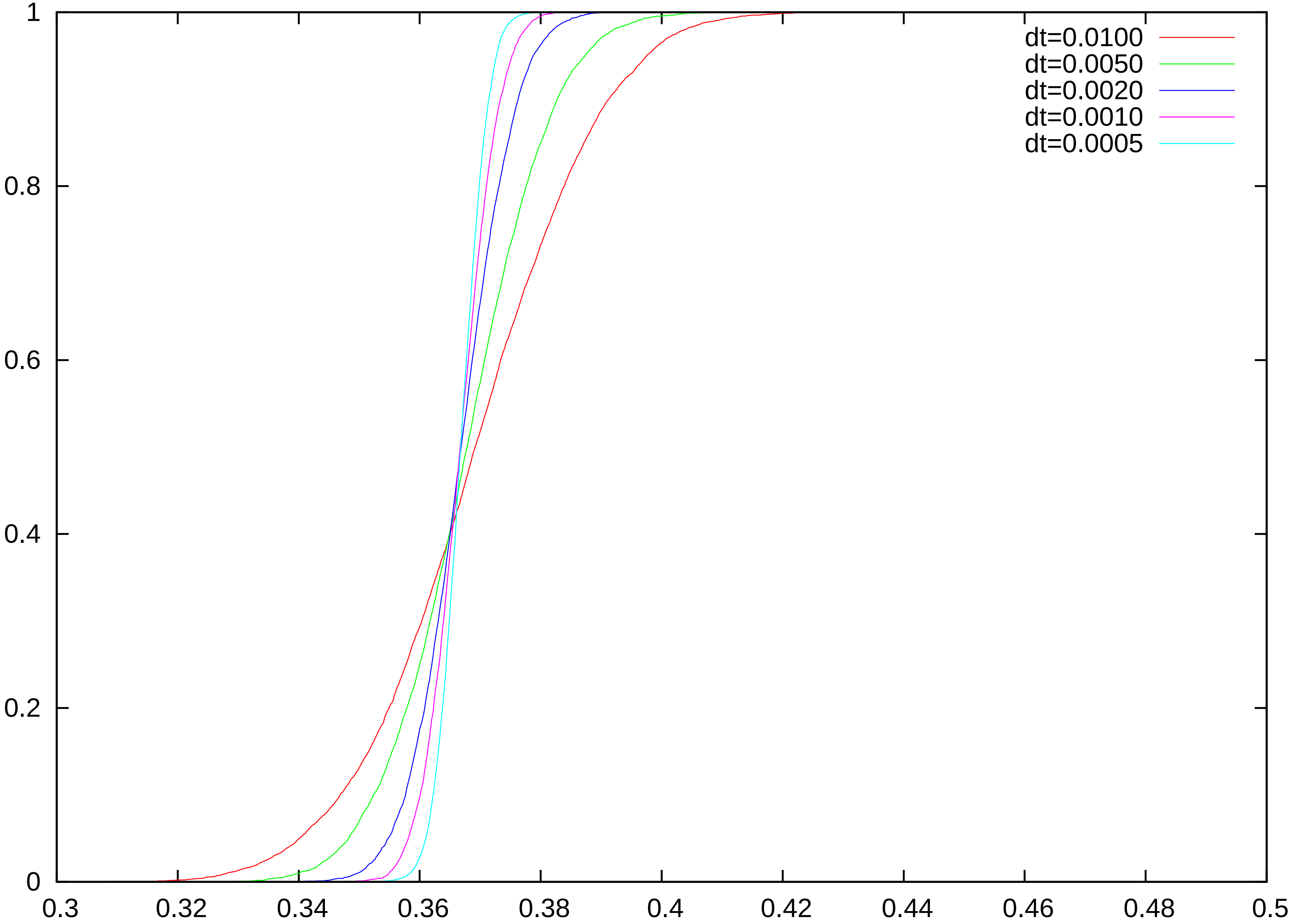
- Theorem of Beffara says SLE curve has Hausdorff dimension $1+\kappa/8$. Show the $1/\nu$ variation of the SLE curve exists with $\nu^{-1} = 1+\kappa/8$.
- Is the $1/\nu$ variation of other discrete models non-random? LSW proved loop-erased random walk converges to SLE$_2$. Simulations indicate $1/\nu$ variation is non-random. ($\nu = 4/5$).

$1/\nu$ variation of LERW at t=0.4,R=4000



Standard deviation of frac. var. vs $\sqrt{dt}$ for LERW