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Abstract of research papers

The numerals refer to the list of publications

Basic notation:
K: a function field of one variable with field of constants $\mathbf{F}_q$.
∞: a place of K.
A: the ring of elements of K integral outside ∞.
Think of K, ∞, A as analogues of $\mathbf{Q}$, ∞ and $\mathbf{Z}$.

(I) Three parts: (i) Gamma functions: $\Gamma : \mathbf{Z}_p \to K_\infty^*$ and interpolations $\Gamma_v : \mathbf{Z}_p \to A_v^*$ at finite places v of K were defined. Functional equations, such as the reflection and multiplication formulae, were proved, by reducing them to manipulations of base q digits of p-adic numbers. The special values of Γ were related to the periods $\tilde{\pi}$ (analogue of $2\pi i$) of Drinfeld modules, e.g. $\Gamma(1/2) = \sqrt{\tilde{\pi}}$. The special values of Γ_v were related to Gauss sums defined in (ii) below, giving elementary proof of an analogue, for $A = \mathbf{F}_q[T]$, of Gross-Koblitz formula, for which only known proofs involve p-adic cohomology.

(ii) Gauss sums: Mixing the classical cyclotomic theory and Drinfeld's cyclotomic theory, an analogue of the Gauss sum, taking values in function fields, was defined. Analogues of classical theorems such as Hasse-Davenport theorem, Stickelberger theorem, Gross-Koblitz theorem, Weil's theorem on Jacobi sums as Hecke characters etc. were proved, with different kind of proofs.

(iii) Zeta functions: Using approximation methods, irrationality and transcendence results for $\zeta(k)$ and $\zeta(k)/\tilde{\pi}^k$, for some special odd k's were proved for the first time; where for $k \in \mathbf{N}$,

$$\zeta(k) = \sum_{n \text{ monic} \in \mathbf{F}_q[T]} \frac{1}{n^k} \in \mathbf{F}_q((1/T))$$

(II) Contains essentially the 'Gauss sum' part of the thesis, but with many proofs different than in (I).

(III) It is shown that $\zeta(k)$ is logarithm of algebraic point on the k-th tensor power (defined by Anderson) of Carlitz module (analogue of motive $\mathbf{Z}(k)$). In particular, using analogues of classical Hermite-Lindemann, Gelfond-Schneider transcendence results on logarithms, which were then proved by Jing Yu, it follows that

Theorem : $\zeta(k)$ is transcendental for all k and $\zeta(k)/\tilde{\pi}^k$ is transcendental for k odd.

There is also a v-adic version for this. Note that for k even, Carlitz had already proved analogue of Euler's theorem that $\zeta(k)/\tilde{\pi}^k$ is rational. Also note that $\zeta(1)$ is analogue of Euler's gamma constant.

(IV) Goss has studied zeta values at negative integers, defined as $\zeta(-k) := \sum_{i=0}^{\infty}(\sum n^k) \in \mathbf{F}_q[T]$, where n runs through monic polynomials of degree i. In this paper, the divided power series corresponding to the measure μ on A_v such that $\int_{A_v} x^k d\mu = \zeta(-k)$, was calculated. The answer turns out to be simple, but quite different than its classical counterpart, suggesting some hidden phenomenon.

(V) Two gamma functions are considered, one defined in the thesis with domain in characteristic zero and the other with domain in characteristic p. These are interpolated, their functional equations, algebraicity and transcendence questions for special values are studied. Their arithmetic is related to the classical cyclotomic theory and Drinfeld's cyclotomic theory resp. Some general conjectures such as Chowla-Selberg phenomenon are given with partial evidence.

(VI) In the thesis, an analogue of the Gauss sums was defined for general A. In (II), it was shown that the Gauss sums for $A = \mathbf{F}_q[T]$ satisfied analogues of many classical theorems. Here it is shown that even though the cyclotomic theory for general A is quite similar, the behavior of Gauss sums is wildly different. For example, classically and in the case $A = \mathbf{F}_q[T]$, the norm of the Gauss sum made up from P-torsion is essentially P. But e. g. for $A = \mathbf{F}_2[x,y]/y^2 + y = x^5 + x^3 + 1$ it is $P(P^\sigma)^2 P^{\sigma^2}$, where σ is automorphism of A given by $\sigma(x) = x + 1$, $\sigma(y) = y + x^2$. Similar behavior holds for all class number one cases.

(VII) Carlitz defined a good analogue $e(z)$ of the exponential function e^z (later greatly generalized by Drinfeld). It is shown here that $e(z)$ and $e = e(1)$ have very interesting continued fractions, though quite different from the classical pattern. Classically, we have Euler's formula

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \cdots]$$

Here e.g. for $q = 2$ and with $[n] = T^{2^n} - T$, we have 'doubling pattern':

$$e = [1, [1], [2], [1], [3], [1], [2], [1], [4], \cdots, [5], \cdots]$$

Interesting continued fractions are also proved for analogues of π, Euler's gamma constant etc.

(VIII) Arithmetically interesting quantities such as factorial and gamma functions, binomial coefficients, zeta functions, in the context of function fields, are related to quantities connected with Drinfeld modules. This is applied to obtain algebraicity of Drinfeld exponentials of some special zeta values. The transcendence properties of exponential, then imply transcendence of such values and their ratios with periods $\tilde{\pi}$. This generalizes (III) in some instances of higher genus. The connection between zeta values and multilogarithms in (III) and here is quite different in spirit form the relations between the classical relative zeta functions and multilogarithms recently investigated by Zagier.

Exa. $A = \mathbf{F}_4[x, y]/y^2 + y = x^3 + \zeta_3$. Then $exp(\zeta(1)) = x^8 + x^4 + x^2 + x$. Note $\zeta(1)$ is analogue of Euler's gamma constant.

(IX) Jacobi sums, in the context of function fields, introduced and studied by combining two types of cyclotomic theories in (I), (II) are shown to be related to Shtukas studied by Drinfeld and theta divisors. This is applied to obtain some results on the factorizations of the analogues of Gauss sums and prove an analogue of Gross-Koblitz formula in general case, generalizing $A = \mathbf{F}_q[T]$ case of (II). For this purpose, a different generalization of the gamma function in $\mathbf{F}_q[T]$ case is introduced and interpolated, at finite as well as infinite places.

(X) The valuations of the function field Gauss sums at the infinite places are shown to be related to Weierstrass gaps. This generalizes the result of (I) for $\mathbf{F}_q[T]$, where the valuations are all $-1/(q-1)$, in direct analogy with the well-known classical result that all the absolute values of Gauss sums are $q^{1/2}$. The additional twist in the general case is existence of finitely many exceptional primes. The sign of quadratic and higher order Gauss sums is determined, giving results in the direction of Gauss' sign theorem and the work of Cassels and Matthews.

(XI) We introduce and study analogues of hypergeometric functions in the settings of function fields over finite fields. We show analogues of differential equations, integral representations, transformation formulas, continued fractions and show that analogues of various special functions and orthogonal polynomials occur as specializations. There are two analogues: one with characteristic zero domain and one with characteristic p domain.

(XII) Orders of vanishing and leading terms of the special zeta values are known to contain deep arithmetic information. This is true also for the characteristic p zeta functions introduced by Carlitz and Goss. Goss showed that these vanish to at least to orders expected by analogies and raised a question whether they are exact. We show that there can be extra vanishing. The pattern involving q-digits that control these extra vanishings gives us a challenge to put these zeros in a general framework. Also, the results have strong implications for the Drinfeld's cyclotomic theory.

(XIII) Developing ideas of (VII) further, we show that analogues of Hurwitz numbers $(ae^{2/n}+b)/(ce^{2/n}+d)$, with a, b, c, d, n integers have continued fractions with very interesting patterns involving block reversing and block repeating moves, in contrast with the classical arithmetic progressions patterns. Hence even though the patterns and the proofs are completely different, for some reason analogous numbers have interesting patterns. For $q = 2$, situation is highly interesting and analogue of 2 in $2/n$ seems to be t, $t+1$ or t^2+t then, for the reasons of rationality of torsion points of carlitz module. We settle the $q > 2$ case and give suggesting examples for the case $q = 2$.

(XIV) Voloch gave a nice proof of the transcendence of the period of the Tate elliptic curve in finite characteristic, by analyzing the Galois action in Igusa tower. This is analogous to Schneider's classical result or rather to Mahler-Manin conjectures. We give another proof of this result based on the transcendence criterion of Christol involving notions of recognizable sequences and automata.

(XV) Our earlier work connecting function field gamma values to periods has corollaries about transcendence of gamma values at fractions, parallel to Chudnovsky's results about gamma values at fractions with denominators 2, 3, 4, 6. Using automata criterion of Christol, which is a completely different method, we show transcendence of many gamma values and monomials in them, with no restrictions on denominators, but some restrictions on the numerators. There is no parallel theorem in the classical case.

(XVI) Building on Krichever-Drinfeld dictionary, Anderson has introduced and studied an analogue of solitons for rational function field, with applications to the values of gamma and zeta functions and Hecke characters. We give a simple approach which gives algebraic equations for solitons, some concrete examples and information on its divisor, Galois action etc. We discuss how the generalization of our approach to general function field differs from the generalization to Anderson's approach and arithmetic applications of both cases.

(XVII) We settle (most of) the remaining (and most interesting) case of $q = 2$ of (XIII), by proving an universal inductive scheme of block repetition and reversal patterns. The patterns depend subtly on prime factorization of one of the parameters of the Hurwitz family.

(XVIII) Using the techniques of Automata theory, we give another proof of the function field analogue of Mahler-Manin conjecture and prove transcendence results for the power series associated to higher divisor functions $\sigma_k(n) = \sum d|nd^k$ and discuss algebraic dependence relations between such series.

(XIX) In this paper, we develop the theory of hypergeometric functions introduced in (XI) further by explaining the noncommutative aspects and various analogies satisfied by it. In particular, we give analogues of Kummer solutions at ∞, by defining a suitable analogue of $(a)_{-n}$. The analogies do not always work as expected naively.

(XX) We propose a computational classification of finite characteristic numbers (Laurent series with coefficients in a finite field) and prove that some classes have good algebraic properties. This provides tools from the theories of computation, formal languages and formal logic for finer study of transcendence and algebraic independence questions. Using them, we place some well-known transcendental numbers occurring in number theory in the computational hierarchy.

(XXI) For each rational diophantine approximation exponent in a certain range, we provide an explicit family of continued fractions of algebraic power series in finite characteristic (together with the algebraic equations they satisfy) which have that exponent. We can take exponent arbitrarily near 2. We also provide some non-quadratic examples with bounded sequence of partial quotients.

(XXII) We give an alternate proof of special case of Anderson's result on log-algebraicity for $F_q[T]$ (with ramifications to new 'cyclotomic unit modules') and provide some new formulas.

(XXIII) It is well-known that while the analogue of Liouville's theorem on diophantine approximation holds in finite characteristic, the analogue of Roth's theorem fails quite badly. We associate certain curves over function fields to given algebraic power series and show that bounds on the rank of Kodaira-Spencer map of this curves imply bounds on the exponents of the power series, with more 'generic' curves (in the deformation sense) giving lower exponents. If we transport Vojta's conjecture on height inequality to finite characteristic by modifying it by adding suitable deformation theoretic condition, then we see that the numbers giving rise to general curves approach Roth's bound. We also prove a hierarchy of exponent bounds for approximation by algebraic quantities of bounded degree.

(XIV) We develop multizeta values in function field arithemtic. There are complex and characteristic p versions. We investigate identities between them and relations with periods, interpolations.

Books

(1) See the table of contents on the web page.
(2) See the table of contents on the web page.

Brief description of Conference proceedings papers:

(1) This paper describes some of the results of the thesis (I). Highlights of part of the thesis containing first transcendence results on the Carlitz zeta function have not been published anywhere else. (Since they were soon improved by (III)).

(2) This is an extract of the different kind of proof for analogue of Gross-Koblitz formula in the thesis (I), together with some related background.

(3) In these lectures for students and teachers, we have tried to motivate and explain various analogies, in spite of major differences, between arithmetic of numbers and polynomials.

(4) This is an expository article on the work on gamma functions in function field. It also contains announcement of results of (IX) and also explains another gamma function.

(5) We have described (sketches of proofs) and put in perspective of Drinfeld's theory, some theorems and conjectures relating class numbers and zeta values at positive and negative integers (there are two distinct theories in contrast to the classical case), analogues of results and conjectures of Kummer and Vandiver, growth rates of class numbers, zeta measures and other aspects of Iwasawa theory.

(6) We describe the automata method, with examples and short sketches of proofs, and explain the two applications to number theory that we have made, namely (XIV) and (XV). Finally, we prove a new result on the transcendence of some values of interpolations of function field gamma function at finite primes.

(7) We describe the emerging theory of L-functions and modular forms in the setting of function fields over finite fields. We give a quick introduction to Drinfeld modules and higher dimensional motives, to arithmetic and analytic properties of finite characteristic zeta functions, modular forms etc.

(8) We give an overview of applications of the concepts and techniques of the theory of integrable systems to the number theory in finite characteristic. The applications include explicit class field theory and Langlands conjectures, how geometry of theta divisor controls factorization of analogues of Gauss sums, special values of Gamma, Zeta and L-functions, analogues of Weil's and Stickelberger's theorems and control of intersection of the Jacobian torsion with the theta divisor. The techniques are Krichever-Drinfeld dictionary and theory of solitons, Akhiezer-Baker and tau functions developed in this context by Greg Anderson.

(9) In contrast to Roth's theorem that all algebraic irrational real num-

bers have approximation exponent two, the distribution of the exponents for the function field counterparts is not even conjecturally understood. We describe some recent progress made on this issue. An explicit continued fraction is not known even for a single non-quadratic algebraic real number. We provide many families of explicit continued fractions, equations and exponents for non-quadratic algebraic laurent series in finite characteristic, including non-Riccati examples with both bounded or unbounded sequences of partial quotients.

(10) We describe the theory of elliptic curves and Drinfeld modules in the function field setting. Both these objects share some of the properties of the elliptic curves familiar in the number fields setting. We develop the basics, describe analogies, give examples, survey and compare the main results and some open questions.

Expository articles

(i) This is a brief account (at the occasion of Fields medal award) of Drinfeld's work on quantum groups, Drinfeld modules and Langlands conjectures and other topics.

(ii) These articles deal with many issues of basics of algebraic number theory and cyclotomic theory in particular, and give many examples, proofs and comments.

Abstracts of PhD thesis of students

(1) Javier Diaz-Vargas (1996) — On zeros of characteristic p zeta functions: We present a simpler proof of Wan's theorem that 'non-trivial' zeros of characteristic p zeta function of Goss interpolating Carlitz zeta function for $F_p[t]$ satisfies Goss' analog of the Riemann hypothesis that they all lie on a 'real line' in 'complex plane' and give partial results for $F_q[t]$, for q not prime. For trivial zeros for Goss zeta functions for general function fields, we give many examples of extra vanishing than that suggested by naive analogies. We give applications to non-vanishing of certain class group components for cyclotomic function fields. We give examples of function fields, where all primes of degree more than two are 'irregular' in the sense of Drinfeld-Hayes cyclotomic theory.

(2) Aaron Ekstrom (1999)— On the infinitude of elliptic carmichael numbers: Under the assumption (much weaker than the current conjectures) that the smallest prime congruent to $-1 \bmod q$ is at most $qexp((logq)^{1-\epsilon})$, for large enough q, we show that there are infinitely many elliptic Carmichael numbers (i.e. composite numbers passing all primality tests for all CM eeliptic curves and points on them). We give many examples, show they are square-free, give bounds on theri number and show that there are no strong elliptic Carmichael numbers.

(3)Justin Miller (2007)— On p-adic continued fractions and quadratic irrationals: Provides new results, computational evidence about known and new (introduced in the thesis) p-adic continued fraction algorithms and a function field analog with regard to questions about convergence, finiteness for rationals, periodicity for quadratic irrationals, symmetry patterns etc. In particular, a new quadratic algorithm introduced seems (numerical evidence, proved only for $p = 2, 3$) to provide periodicity for $p < 37$, improving results of Browkin and a new symmetry different from the real case or other p-adic cases is observed.

Two papers based on (1) are published in Journal of Number Theory.

I have lost contact with Ekstrom after he graduated. His thesis is available on request.