MATLAB PROJECTS, SPRING 2012

MATH 215

1. Topic 1: Balancing Chemical Equations

Written by: Rebecca Strautman

Problem Description:

Chemical equations are written with the reactants (the chemicals you start with) on the left, and the products (the chemicals you end with) on the right. By the Law of Conservation of Mass, the number of atoms must be the same on both sides (for instance, if you have 2 atoms of carbon on the left, you must have 2 atoms of carbon on the right) because these atoms cannot be created or destroyed in a reaction. Balancing an equation means guring out how much of each reactant you need to make how much of each product. For example, to balance the equation:

$$H_2 + O_2 \to H_2O$$

we simply do the following:

$$2H_2 + O_2 \to 2H_2O,$$

where we have 4 atoms of H on each side, and 2 of O, so it is balanced. For many problems, you can probably gure the constants out in your head, but if there are many elements involved, we can instead use linear algebra.

Example Problem:

Consider the unbalanced reaction

$$C_2H_6 + O_2 \to CO_2 + H_2O.$$

Assign each molecule a variable $x_1, \ldots, x_4$ since we have 4 expressions in the reaction:

$$x_1C_2H_6 + x_2O_2 \to x_3CO_2 + x_4H_2O.$$

Now, we write out the requirements for each element

$$\begin{cases} C : & 2x_1 = x_3 \\ H : & 6x_1 = 2x_4 \\ O : & 2x_2 = 2x_3 + x_4. \end{cases}$$

Bringing everything to the left hand side, we have a system of linear equations which we can write in the form

$$\begin{cases} 2x_1 - x_3 & = 0 \\ 6x_1 - 2x_4 & = 0 \\ 2x_2 - 2x_3 - x_4 & = 0. \end{cases}$$

We put these into an augmented matrix and get

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{pmatrix}.$$

The reduced row echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & -1 & -1/3 & 0 \\ 0 & 1 & 0 & -7/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{pmatrix}.$$

This gives the solution set

$$\mathbf{x} = \begin{pmatrix} x_4/3 \\ 7x_4/6 \\ 2x_4/3 \\ x_4 \end{pmatrix}.$$

We need to choose x_4 such that the vector $\mathbf{x}$ contains only integers, and so the components of $\mathbf{x}$ do not have any common factors (we want the equation to be in its reduced form). In this solution set, we can take $x_4 = 6$. So our balanced chemical equation is

$$2C_2H_6 + 7O_2 \to 4CO_2 + 6H_2O.$$

Assigned Problem for Project 1A

Use the method described above to balance each of the following chemical equations:

(a) $K_4Fe(CN)_6 + H_2SO_4 + H_2O \to K_2SO_4 + FeSO_4 + (NH_4)_2SO_4 + CO,$

(b) $MnS + As_2Cr_{10}O_{35} + H_2SO_4 \to HMnO_4 + AsH_3 + CrS_3O_2 + H_2O$

(c) $Fe_2(SO_4)_3 + KSCN \to K_3Fe(SCN)_6 + K_2SO_4$

Assigned Problem for Project 1B

Use the method described above to balance each of the following chemical equations:

(a) $PhCH_3 + KMnO_4 + H_2SO_4 \to PhCOOH + K_2SO_4 + MnSO_4 + H_2O.$

(b) $K_4Fe(CN)_6 + KMnO_4 + H_2SO_4 \to$
$KHSO_4 + Fe_2(SO_4)_3 + MnSO_4 + HNO_3 + CO_2 + H_2O$

(c) $PbN_6 + CrMn_2O_8 \to Pb_3O_4 + Cr_2O_3 + MnO_2 + NO$

A few helpful hints: Subscripts on an atom denote the number of atoms of that element in one molecule of the reactant or product. If multiple atoms are in parentheses with a subscript $(NO_3)_2$, this subscript distributes and you have 2 atoms of nitrogen N and 6 of oxygen O. Finally, for those who are unfamiliar with chemistry, abbreviations for elements consist of either a capital letter O, N, C, ..., or a capital letter followed by a lowercase one Na, Cl, Ag, ..., so KCl consists of two elements, denoted by K and Cl.

2. Topic 2: Cryptography

Written by: Rebecca Strautman

Problem Description:

Sometimes it is necessary to encode messages, for instance, when sensitive information that is sent over the internet (Imagine how 007 would communicate with M via email!). The main idea in cryptography is to encode a message so that the only person who can decode it is someone with a deciphering key. One simple way of encoding messages is through the use of matrices to transform the message. A message, in the form of numbers, can be created, then encoded with a matrix operation acting on these numbers. A key would consist of a decoding matrix which one can use to reproduce the original message.

In its most basic form, one would take a message consisting of letters a through z and assign each a number 1 through 26, encode the message, then send a user the key. Clearly, this can be easily broken in many dierent ways. One can simply guess entries for the *key*, decode the message, and attempt to see if the resulting message makes sense for different combinations of a through z. For small matrices, a computer can easily crack these codes. In the real world, one does not simply use numbers 1 through 26. One may use modular arithmetic, and perhaps assign a through z to be multiple values from the product of two large prime numbers, say 1, 747, 822, 896, 920, 092, 227, 343 and 105, 646, 155, 480, 762, 397 (can you check that they are indeed primes?). In order to decode, one would need to know both of these large primes. It is very dicult to find large primes in general, and using the previous guess and check type method would not work for messages of this form. We will consider the basic case for this project.

Example Problem:

Let us encode the message "`LINEAR ALGEBRA IS FUN.`". First, we must assign a number to each character (letter and punctuation mark). For simplicity's sake, let's set $A = 1$, $B = 2$, ... $Z = 26$. We also need numbers for the space character and the period, so we can choose 27 for space, and 28 for period. Now we can convert each character to a number.

L	I	N	E	A	R		A	L	G	E	B	R	A		I	S		F	U	N	.
12	9	14	5	1	18	27	1	12	7	5	2	18	1	27	9	19	27	6	21	14	28

If we stopped here, the code could be very easily broken, so we need to scramble it. We can use a matrix to do so. We consider the following 3 x 3 encoding matrix:

$$C = \begin{pmatrix} -3 & -3 & 4 \\ 0 & 1 & 1 \\ 4 & 3 & 4 \end{pmatrix}.$$

In order to multiply our message by this matrix, we need to put our message into a matrix with 3 rows. More specically, we need to put our message into a 3 x N-matrix, so that the multiplication is defined. We can do this putting the message into 3 x 1 column vectors:

$$\begin{pmatrix}12\\9\\14\end{pmatrix}, \begin{pmatrix}5\\1\\18\end{pmatrix}, \begin{pmatrix}27\\1\\12\end{pmatrix}, \begin{pmatrix}7\\5\\2\end{pmatrix}, \begin{pmatrix}18\\1\\27\end{pmatrix}, \begin{pmatrix}9\\19\\27\end{pmatrix}, \begin{pmatrix}6\\21\\14\end{pmatrix}, \begin{pmatrix}28\\27\\27\end{pmatrix}$$

Note that since the number of letters in the message is not divisible by 3, we fill in the last column vector with space characters. Now we use these column vectors as the columns of a 3 x 8-matrix:

$$A = \begin{pmatrix}12 & 5 & 27 & 7 & 18 & 9 & 6 & 28\\ 9 & 1 & 1 & 5 & 1 & 19 & 21 & 27\\ 14 & 18 & 12 & 2 & 27 & 27 & 14 & 27\end{pmatrix}.$$

Multiplying the encoding matrix by this matrix gives:

$$CA = \begin{pmatrix}-119 & -90 & -132 & -44 & -165 & -192 & -137 & -273\\ 23 & 19 & 13 & 7 & 28 & 46 & 35 & 54\\ 131 & 95 & 159 & 51 & 183 & 201 & 143 & 301\end{pmatrix}.$$

The encoded message is the list the numbers in order of the column vectors (i.e. -119, 23, 131, -90, 19, 95, -132, 13, ...). To decode this message, we would write these numbers as column vectors, and then multiply by the decoding matrix (the inverse of the encoding matrix C). In our example, we have the decoding matrix of

$$C^{-1} = \begin{pmatrix}1 & 0 & 1\\ 4 & 4 & 3\\ -4 & -3 & -3\end{pmatrix}.$$

Now, by looking at $C^{-1} \cdot CA$ (which is the original A), and with the numbers in order by column vector

$$(12, 9, 14, 5, \ldots, = L, I, N, E, \ldots),$$

we get our original message.

Assigned Problem for Projects 2A, 2B:

Use the previous method to translate a code into the original message. The following was used as the ENCODING matrix:

$$\begin{pmatrix}4 & 3 & 4 & 6 & -2 & -1\\ 8 & 9 & 4 & -2 & -2 & -5\\ -3 & 0 & 3 & -1 & 1 & 2\\ 8 & 9 & 8 & 9 & 8 & -9\\ 4 & 2 & -2 & 3 & 0 & 0\\ 4 & 3 & -1 & -1 & 4 & 5\end{pmatrix}.$$

The encoded code for Project 2A is:

[283, 181, 3, 455, 132, 186; 267, 25, 61, 384, 49, 31; 257, 187, 45, 704, 151, 299; 184, 310, 29, 516, 66, 167; 240, 386, 26, 515, 107, 218; 197, -19, 116, 340, 31, 181; 236, -42, 56, 245, 71, 101; 276, 98, 135, 690, 82, 270; 190, 218, 58, 687, 49, 166; 73, 105, 39, 396, 41, 155; 298, 224, -31, 807, 179, 219; 255, 303, -54, 557, 160, 134; 373, 317, 60, 884, 183, 373]

The encoded code for Project 2B is:

272, 152, -53, 467, 167, 124, 64, -30, 69, 101, 35, 220, 249, 23, 36, 408, 77, 73, 299, 249, 6, 770, 159, 208, 157, 155, 26, 418, 113, 223, 284, 386, 27, 496, 150, 320, 86, -92, 53, 10, 44, 159, 248, 66, 20, 555, 77, 42, 29, -31, 51, 205, 66, 306, 279, 187, -67, 530, 167, 89, 248, 302, -7, 781, 173, 286, 228, 274, 69, 384, 94, 264, 202, -92, 89, 458, 78, 230.

For each of the projects, translate the encoded code back into a message (using the same numbering convention $A = 1$, $B = 2$, etc).

Note: When Matlab does computations, it will do everything in floating point arithmetic. Sometimes there will a condition known as roundoff error in which a solution which would come out exact by hand comes out approximate on a computer. For this problem, you can safely assume that if Matlab returns numbers such as 8.000000000000014 then the answer should really be 8.

3. Topic 3: Genetics

Written by: Rebecca Strautman

Problem Description:

Organisms inherit traits from their parents encoded by segments of DNA called genes. Genes have different forms that cause different characteristics, for example brown eyes vs. blue eyes, or straight hair vs. curly hair. In simple representations, the dominant form is symbolized by a capital letter (A), and the recessive form is symbolized by a lowercase letter (a). An organism has two forms of each gene, one from each parent. This two-letter set is known as its genotype for that gene. For example, ospring from parents with the genotypes AA and aa will have the genotype Aa.

We can model this inheritance as a transition from one generation to another. Each transition is given by some matrix of probabilities that the next generation will have some given properties (for example, an AA and an aa produce an Aa with probability 1). One wishes to know the long term behavior so we take the matrix product over many generations. We'll see that certain genetic traits will die out over time, while others will dominate. This process of transitioning is very closely related to the concept of Markov chains or Markov processes.

Example Problem:

In rats, F (the version of the gene for fur) is dominant to f (the version of the gene for hairless). Since dominant traits override recessive traits, FF rats have fur, Ff rats have fur, and ff rats are hairless. If 1/3 of a population of rats has each of these genotypes, then the distribution of genotypes is given by the vector

$$\mathbf{x}_0 = \begin{pmatrix} FF \\ Ff \\ ff \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Consider crossing these offspring with rats with the genotype FF only. For FF crossed with FF, the probabilities of offspring with FF, Ff, and ff are, respectively, 1, 0, and 0. For FF crossed with Ff, the probabilities of FF, Ff, and ff, are 1/2, 1/2, and 0. For FF crossed with ff, the probabilities of FF, Ff and ff, are 0, 1, and 0.

In the total population of offspring, the probability of the genotype of FF is $(1)(1/3)+(1/2)(1/3)+(0)(1/3) = 1/2$. The probability of Ff is $(0)(1/3)+(1/2)(1/3)+(1)(1/3) = 1/2$. The probability of ff is $(0)(1/3) + (0)(1/3) + (0)(1/3) = 0$. So the vector $\mathbf{x}_1$, representing the probabilities of each genotype in the first generation, can be given by $\mathbf{x}_1 = A\mathbf{x}_0$, where

$$A = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that our transition matrix, in the first row, gives the probability of that FF crossed with FF, Ff and ff is an FF. The second row gives the probability that FF crossed with FF, Ff and ff is an Ff, and the third row gives the probability that FF crossed with FF, Ff and ff is ff. Since we want to see what happens after multiple generations, we treat this as an iteration $\mathbf{x}_k = A\mathbf{x}_{k-1}$ which makes $\mathbf{x}_k = A^k\mathbf{x}_0$. We find

$$\mathbf{x}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \ \mathbf{x}_1 = \begin{pmatrix} 3/4 \\ 1/4 \\ 0 \end{pmatrix}, \ \mathbf{x}_1 = \begin{pmatrix} 7/8 \\ 1/8 \\ 0 \end{pmatrix}, \ \mathbf{x}_1 = \begin{pmatrix} 15/16 \\ 1/16 \\ 0 \end{pmatrix}, \ \mathbf{x}_1 = \begin{pmatrix} 31/32 \\ 1/32 \\ 0 \end{pmatrix}, \ \ldots$$

These are converging to the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. This means that eventually, assuming there are no mutations, the rats in this population will no longer carry the version of the gene for hairlessness.

Assigned Problem:

Consider, now, a plant with a gene that codes for flower color (R for red, and r for white, where R is dominant), and a gene that codes for height (T for tall, and t for short, where T is dominant). These genes are on dierent chromosomes, and are inherited independently of each other. So the dierent combinations of genes are $RRTT$, $RRTt$, $RRtt$, $RrTT$, …, etc. The probabilities of each genotype are given by the vector

$$\mathbf{x}_0 = \begin{pmatrix} RRTT \\ RRTt \\ RRtt \\ RrTT \\ RrTt \\ Rrtt \\ rrTT \\ rrTt \\ rrtt \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

Suppose we cross these plants with only plants that have the genotype $rrTt$. What is the probability for each genotype in the 6th generation? The 12th ? The 40th ? What genotype(s) does the population seem to be converging to? What do these plants look like?

4. Topic 4: Orbits

Written by: Rebecca Strautman

Problem Description:

Given two points in a plane, we know how to find a line going through them by solving a system of linear equations

$$\begin{cases} ax_1 + by_1 + c = 0 \\ ax_2 + by_2 + c = 0. \end{cases}$$

Similarly, given three points in a plane, we can find a circle going through them by examining a system of equations. If (x, y) is a point on a circle, then we know that for some constants a, b, and c, we have

$$a(x^2 + y^2) + bx + cy + d = 0.$$

Taking multiple points on a circle, we have a system such as

$$\begin{cases} a(x^2 + y^2) + bx + cy + d &= 0 \\ a(x_1^2 + y_1^2) + bx_1 + cy_1 + d &= 0 \\ a(x_2^2 + y_2^2) + bx_2 + cy_2 + d &= 0 \\ a(x_3^2 + y_3^2) + bx_3 + cy_3 + d &= 0 \end{cases}.$$

We cannot simply put this system into a matrix and solve as we normally do, because these are not linear equations, but we do have some useful information. Since there are innitely many points on a circle, there are infinitely many solutions to this system (if a, b, c and d solve the equations then ka, kb, kc and kd also solve them for any $k \neq 0$). This means that the determinant of the matrix must be 0. We have

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Recalling properties of the determinant, we then have

$$0 = \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} =$$

$$= (x^2+y^2)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - x\begin{vmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{vmatrix} + y\begin{vmatrix} x_1^2 + y_1^2 & x_1 & 1 \\ x_2^2 + y_2^2 & x_2 & 1 \\ x_3^2 + y_3^2 & x_3 & 1 \end{vmatrix} - \begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 \\ x_2^2 + y_2^2 & x_2 & y_2 \\ x_3^2 + y_3^2 & x_3 & y_3 \end{vmatrix}.$$

Since the determinant of each of these matrices in the right-hand-side is just a number, we have the equation for a circle.

Example Problem:

Let us consider a circle that contains the points $(0,0)$, $(4,5)$ and $(9,2)$. We can plug in these points to obtain:

$$0 = \begin{vmatrix} x^2+y^2 & x & y & 1 \\ 0 & 0 & 0 & 1 \\ 41 & 4 & 5 & 1 \\ 85 & 9 & 2 & 1 \end{vmatrix} = -37(x^2+y^2) + 343x + 29y.$$

A nice way to reduce the complexity of this problem in Matlab would be to program the lower rows of the system as a matrix. For example, we write

$$A = [0,0,0,1;41,4,5,1;85,9,2,1]$$

Then we can simply tell Matlab to compute the determinant of each submatrix:

$$det(A(:,[2\ 3\ 4])),\ det(A(:,[1\ 3\ 4])),\ det(A(:,[1\ 2\ 4])),\ det(A(:,[1\ 2\ 3])).$$

Here $det(A(:,[2\ 3\ 4]))$ asks Matlab to compute the determinant using the 2nd, 3rd, and 4th columns. The other determinants use the same idea.

Assigned Problem:

A similar technique can be used to find an equation approximating the orbit of an asteroid. If an astronomer observes an asteroid at five different times, and records the coordinates, this technique can be used to find an equation for the orbit. Suppose the coordinates recorded are:

$$(6.02, 5.44),\ (7.23, 3.01),\ (8.44, 0.94),\ (10.03, -1.37),\ (12.15, -3.72).$$

Use the following equation for an ellipse:

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6 = 0.$$

5. Topic 5: Two-dimensional Diffusion

Written by: Paul Dostert

Problem Description:

Let us consider a rectangular plate with some heat sources on each edge. Assuming these heat sources remain constant, the plate will eventually reach an equilibrium temperature. You can think of this, in 3D, as a cubic heat sink, perhaps on a CPU. The CPU is releasing what we will assume is a constant heat source, while the ambient air is staying at a constant temperature. This would make 1 side of the cube a very high temperature (from the CPU) and the other 5 sides a relatively low temperature (room temperature). We are, of course, ignoring any fans or buildup of heat within the enclosure. The 2D analog to this problem would be some cross section cut out from this cube.

We make the assumption that there is an underlying grid on our rectangular plate. To determine the equilibrium, we may assume that if the plate is at equilibrium and xi is a grid point not on the boundary, then the temperature at x_i is given by the average of the temperatures of the four closest grid points to x_i . This creates a linear system for x_i , which we can then solve using the linear algebra techniques learned in class. This problem actually corresponds to solving the partial differential equation

$$\Delta u(x, y) = 0,\ 0 < x, y < 1$$

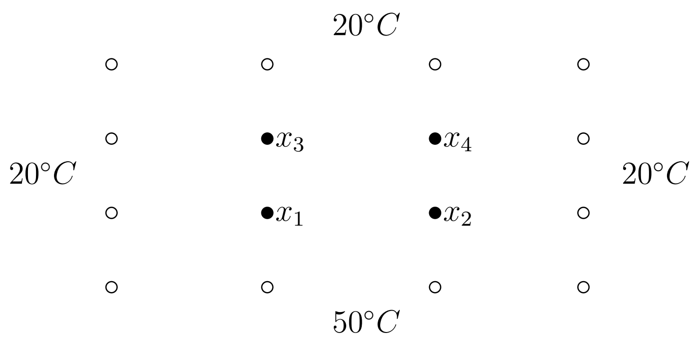
with boundary conditions

$$u(x, 0) = u_0,\ u(x, 1) = u_1,\ u(0, y) = u_2,\ u(1, y) = u_3$$

using finite differences in two dimensions. (Do not worry if if you have not seen them before.)

Example Problem:

We consider a rectangular plate with temperature $50°C$ on the bottom boundary and $20°C$ on the other three boundaries. We create a uniform grid with 4 interior grid points, and wish to solve for the temperature at each x_i , which we will denote by u_i, $i = 1, \ldots, 4$. The problem is illustrated in the graphic below:



The idea is that we must determine equations for each u_i. Let us start with u_1. We know that u_1 will be given by the average of the 4 temperatures around it, thus

$$u_1 = \frac{20 + 50 + u_2 + u_3}{4} .$$

Using the same idea, we have

$$u_2 = \frac{20 + 50 + u_1 + u_4}{4} , \ u_3 = \frac{20 + 20 + u_1 + u_4}{4} , \ u_4 = \frac{20 + 20 + u_2 + u_3}{4} .$$

Moving all of the constants to the right-hand-side and all of the unknowns to the left, we have the following system

$$\begin{cases} 4u_1 - u_2 - u_3 & = 70 \\ -u_1 + 4u_2 - u4 & = 70 \\ -u_1 + 4u_3 - u_4 & = 40 \\ -u_2 - u_3 + 4u_4 & = 40 \end{cases}$$

which has the following augmented matrix

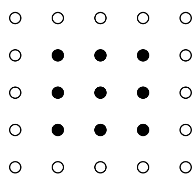
$$\begin{pmatrix} 4 & -1 & -1 & 0 & 70 \\ -1 & 4 & 0 & -1 & 70 \\ -1 & 0 & 4 & -1 & 40 \\ 0 & -1 & -1 & 4 & 40 \end{pmatrix} .$$

When solved, we have

$$\mathbf{u} = \begin{pmatrix} 31.25 \\ 31.25 \\ 23.75 \\ 23.75 \end{pmatrix} .$$

Assigned Problem:

Consider a rectangular plate with temperature $70°C$ on the bottom boundary and $25°C$ on the other three boundaries. Create a uniform grid with 9 interior grid points, 3 in each direction:



Using the same idea as the previous example, derive a matrix A and a right-hand-side $\mathbf{b}$ so that the solution to $A\mathbf{u} = \mathbf{b}$ returns the temperatures given by $\mathbf{u} = [u_1, u_2, ..., u_9]^T$. Solve the system using Matlab. Comment on what A looks like (but do NOT print out A) if we have 16 interior points (4 x 4). What about if we have 64 interior points?

6. Topic 6: Directed Graphs

Written by: Hermann Flaschka

Problem Description:

A directed graph is a graph some or all of whose edges are equipped with arrows. We consider only graphs with **no** multiples edges (so any two vertices can be connected by at most one edge), and **no** loops (so no edge can go from a vertex to the same vertex). Also, we will assume our graphs are **connected**, that is, given any two vertices, there is a chain of edges that connect them.

A **walk** on the directed graph may traverse an edge only in the direction of the arrow, if there is one; if an edge does not have an arrow, it can be traversed in both directions. Figure 1 shows a directed graph. The steps $1 \to 3$ and $3 \to 1$ are both permitted, but of $1 \to 2$ and $2 \to 1$, only the former is legal. The 3-step walks $2 \to 3 \to 4 \to 2$ is allowed, whereas the 3-step walk $2 \to 4 \to 3 \to 2$ is not allowed.

The adjacency matrix A of a directed graph is defined as follows. $A_{ij} = 1$ if there is an edge connecting vertex i with vertex j, and either the edge has no arrow, or it has an arrow pointing from i to j. $A_{ij} = 0$ if there is no edge connecting vertex i and vertex j, or if an arrow points from j to i rather than from i to j.

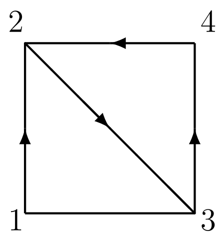


Figure 1

The adjacency matrix of the directed graph in Figure 1 is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Enter this matrix into Matlab. Have it compute A^2 and A^3. You should get

$$A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The (3, 3)-entry of A^3 is 2. Check that there are indeed two legal 3-step walks from vertex 3 to vertex 3 ("legal" means: they never go against the direction of an arrow).

On the other hand, the fact that the (2,4)-entry of A^3 is zero means that there are no legal 3-step walks from vertex 2 to vertex 4 (there is a legal 2-step walk, though).

Assigned Problem: Consider the directed graph in Figure 2.

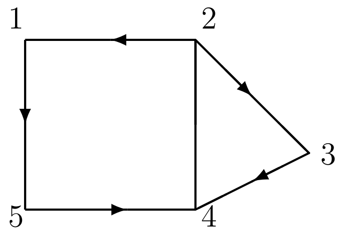


FIGURE 2

a) Find the adjacency matrix A of this directed graph.

b) Using Matlab, show that, for this directed graph, there is an integer m with the following properties.

(i) One can get from any vertex to any other vertex by a legal m-step walk.

(ii) However, if $k < m$, then there is at least one pair of vertices i and j that cannot be connected by a legal k-step walk starting at i and ending at j.

Illustration: (the numbers are made up). Suppose $m = 5$. Then statement (i) says, in particular, that one can start at vertex 1 and legally walk to vertex 3 in five steps, and that one can start at vertex 3 and legally walk to vertex 1 in five steps. But in four steps, maybe one can't legally get from vertex 1 to vertex 5; in three steps, maybe one can't legally get from vertex 4 to vertex 1, etc.

c) In part b), you found that one could start at any vertex and (legally) reach any other vertex in m steps. Now take a number $n > m$. It is a fact that one can start at any vertex and (legally) reach any other vertex in n steps. Explain why this is true. (This property is a consequence of the connection between numbers of n-step walks and the n-th power A^n, of the adjacency matrix. It would be quite hard to deduce from an analysis of the graph.)

d) Now suppose that we reverse the arrow of the edge connecting vertices 3 and 4 (so the arrow in the new graph points from 4 to 3). Does the integer m in b) still exist? Explain your answer.