Generalized Functions in Minkowski Space

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1. Introduction

Periodic functions are basic to many areas of mathematics and physics. The fundamental examples of such functions are the familiar sine and cosine of trigonometry. Our purpose here is to generalize the trigonometry functions. I follow David Shelupsky's idea [1] to introduce the alpha and beta functions. These functions are the analogues of sine and cosine in Minkowski space. i.e. We replace the Pythagorean theorem $x^2 + y^2 = r^2$ with $x^s + y^s = r^s$ for some integer s. I am interested in transforming the Laplace's equation into polar coordinate of Minkowski Space. The calculation leads to a partial differential equation that, when $s = 2$, reduces to Laplace's equation. We use separation of variables to find particular solution to this equation.

2. Generalized Trigonometric Functions

The sine and cosine functions are periodic functions with period 2π. Our task in this section is to extend the class of trigonometric functions.

I will first give a brief introduction to periodic functions [6] and discuss the differential system that describes sine and cosine. After that, I will follow Shelupsky's method [1] to generate the alpha and beta functions.

2.1. Periodic Functions

Periodic functions occur frequently. Think of clocks, sounds, the days and nights, etc. By periodic we mean something that repeats its motion in a constant length of time. Hence, a function $f(t)$ is periodic if

$$\exists T \neq 0 \ni f(t+T) = f(t). \tag{2.1.1}$$

To make a more formal mathematical definition, for any function $f : \Re \to \Re$, let

$$P(f) = \{T \in \Re : f(t+T) = f(t)\}. \tag{2.1.2}$$

Clearly $P(f)$ contains zero for any function. Now we can define $f(t)$ be a periodic function if there exists a non-zero element in $P(f)$. On the other hand, any non-zero element T in $P(f)$ is called a periodic of $f(t)$. If there exists a smallest non-zero element T in $P(f)$, then T is often called the fundamental period of $f(t)$.

2.2. The Differential System Description of Sine and Cosine

Like many other special functions, the sine and cosine can be defined by differential system. Let's consider two functions $\alpha(t)$ and $\beta(t)$. Then the differential system

$$\begin{cases} \dfrac{d}{dt}\alpha(t) = \beta(t) \\ \dfrac{d}{dt}\beta(t) = -\alpha(t) \end{cases} \tag{2.2.1}$$

with the initial condition $\alpha(0) = 0, \beta(0) = 1$, has a unique solution $\alpha(t) = \sin(t)$ and $\beta(t) = \cos(t)$.

2.3. The Alpha and Beta Functions

The idea of the generalizing the functions into Minkowski space starts from generalizing the differential system (2.2.1). Consider a natural number s, we write

$$\begin{cases} \frac{d}{dt}\alpha_s(t) = \beta_s^{s-1}(t) \\ \frac{d}{dt}\beta_s(t) = -\alpha_s^{s-1}(t) \end{cases} \qquad (2.3.1)$$
$$\alpha_s(0) = 0, \beta_s(0) = 1$$

When $s = 2$, (2.3.1) reduce to (2.2.1) so we have $\alpha_2(t) = \sin t$ and $\beta_2(t) = \cos t$. Anyway, we will see $\alpha_s(t)$ and $\beta_s(t)$ have many interesting properties similar to sine and cosine.

Multiplying the two equations in (2.3.1) by $\alpha_s^{s-1}(t)$ and $-\beta_s^{s-1}(t)$, we obtain

$$\begin{cases} \alpha_s^{s-1}(t)\frac{d}{dt}\alpha_s(t) = \alpha_s^{s-1}(t)\beta_s^{s-1}(t) \\ -\beta_s^{s-1}(t)\frac{d}{dt}\beta_s(t) = \alpha_s^{s-1}(t)\beta_s^{s-1}(t) \end{cases}. \qquad (2.3.2)$$

The left hand sides of (2.3.2) are equal, hence

$$\alpha_s^{s-1}(t)\frac{d}{dt}\alpha_s(t) = -\beta_s^{s-1}(t)\frac{d}{dt}\beta_s(t). \qquad (2.3.3)$$

$$\alpha_s^{s-1}(t)\frac{d}{dt}\alpha_s(t) + \beta_s^{s-1}(t)\frac{d}{dt}\beta_s(t) = 0. \qquad (2.3.4)$$

Integrating (2.3.4), we conclude for some constant C,

$$\alpha_s^s(t) + \beta_s^s(t) = C. \qquad (2.3.5)$$

Set $t = 0$ and apply the initial conditions in (2.3.1), the constant is equal to one. Hence,

$$\alpha_s^s(t) + \beta_s^s(t) = 1 \qquad (2.3.6)$$

Note that when $s = 2$, this is the familiar $\sin^2(t) + \cos^2(t) = 1$.

2.4. The Inverse Alpha and Beta Functions

The behavior of $\alpha_s(t)$ and $\beta_s(t)$ are not totally understood. However, their inverse functions have very nice expressions. We will use the notation $\arg$ to represent the inverse function. Let $x = \alpha_s(t)$. With (2.3.1) and (2.3.6) we obtain

$$\frac{dx}{dt} = \beta_s^{s-1}(t) = \left(1 - x^s\right)^{\frac{s-1}{s}} . \qquad (2.4.1)$$

Solve by separating variables, we get

$$t = \int \left(1 - x^s\right)^{\frac{1-s}{s}} dx + C \quad . \qquad (2.4.2)$$

By the fundamental law of calculus and the definition of $\alpha_s(t)$, we conclude the inverse function of $\alpha_s(t)$ is

$$\arg \alpha_s(x) = \int_0^x \left(1 - x'^s\right)^{\frac{1-s}{s}} dx' \quad . \qquad (2.4.3)$$

Similarly, let $x = \beta_s(t)$ so that $dx/dt = -\alpha_s^{s-1}(t)$. Then we have

$$\frac{dx}{dt} = -\left(1 - x^s\right)^{\frac{s-1}{s}} . \qquad (2.4.4)$$

$$t = -\int \left(1 - x^s\right)^{\frac{1-s}{s}} dx + C \quad . \qquad (2.4.5)$$

Again we apply the fundamental law of calculus and the definition of $\beta_s(t)$,

$$\arg \beta_s(x) = \int_x^1 \left(1 - x'^s\right)^{\frac{1-s}{s}} dx' \quad . \qquad (2.4.6)$$

An interesting property of $\alpha_s(t)$ and $\beta_s(t)$ can be obtained here. Suppose s is even. Then the integrand of $\arg \alpha_s(x)$ (2.4.3) is even so that the integral of the upper limit is an odd function. Therefore, $\alpha_s(t)$ is an odd function since its inverse is odd.

Moreover, since $d\beta_s(t)/dt = \alpha_s^{s-1}(t)$ is odd because s is even and $\alpha_s(t)$ is odd. We conclude that $\beta_s(t)$ is even functions.

3. Generalized Pi and Minkowski Space

Everybody knows what is π, but rarely able to expend why π has its value. In fact, it is well known that we can get a series expression of π from the inverse tangent. In this section, we define the generalized Pi by the inverse alpha function or inverse beta function; analyze the properties of this number.

3.1. The Generalized Pi

Recall in the trigonometric functions, we have $\arg\sin(1) = \arg\cos(0) = \pi/2$. We follow this and define a number π_s by the relation

$$\frac{\pi_s}{2} = \arg\alpha_s(1) = \arg\beta_s(0) = \int_0^1 \left(1 - x^s\right)^{\frac{1-s}{s}} dx . \qquad (3.1.1)$$

Then π_s is known as the generalized Pi. From the relation (3.1.1) it is clear that

$$\begin{cases} \alpha_s\left(\dfrac{\pi_s}{2}\right) = 1 \\ \beta_s\left(\dfrac{\pi_s}{2}\right) = 0 \end{cases} . \qquad (3.1.2)$$

Now we have $\alpha_s(t)$ and $\beta_s(t)$ defined in the interval $0 \le t \le \pi_s/2$. However, we've shown that $\alpha_s(t)$ is odd function and $\beta_s(t)$ is even function when s is even, so the domain of $\alpha_s(t)$ and $\beta_s(t)$ are defined in the interval $-\pi_s/2 \le t \le \pi_s/2$ for even s.

3.2. The Periodic Alpha and Beta

In this section we want to show when s is even, $\alpha_s(t)$ and $\beta_s(t)$ are periodic function with period $2\pi_s$. We will start by showing

$$\begin{cases} \alpha_s\left(\dfrac{\pi_s}{2} - t\right) = \beta_s(t) \\ \beta_s\left(\dfrac{\pi_s}{2} - t\right) = \alpha_s(t) \end{cases} . \qquad (3.2.1)$$

To prove (3.2.1) is true, we write the right hand side of (3.1.1) as

$$\int_0^1 \left(1-x^s\right)^{\frac{1-s}{s}} dx = \int_0^x \left(1-x'^s\right)^{\frac{1-s}{s}} dx' + \int_x^1 \left(1-x'^s\right)^{\frac{1-s}{s}} dx'. \tag{3.2.2}$$

Which is the same as

$$\frac{\pi_s}{2} = \arg\alpha_s(x) + \arg\beta_s(x). \tag{3.2.3}$$

We take $\arg\beta_s(x)$ to the let and put both sides into the alpha function, then

$$x = \alpha_s\left(\frac{\pi_s}{2} - \arg\beta_s(x)\right). \tag{3.2.4}$$

Let $t = \arg\beta_s(x)$, we have

$$\alpha_s\left(\frac{\pi_s}{2} - t\right) = \beta_s(t). \tag{3.2.5}$$

Which is the first part of (3.2.1). Similarly, from equation (3.2.3) we take $\arg\alpha_s(x)$ to the left and put both sides into the beta function, then

$$x = \beta_s\left(\frac{\pi_s}{2} - \arg\alpha_s(x)\right). \tag{3.2.6}$$

Let $t = \arg\alpha_s(x)$, then

$$\beta_s\left(\frac{\pi_s}{2} - t\right) = \alpha_s(t). \tag{3.2.7}$$

Which is the second part of (3.2.1).

We consider the function $\alpha_s(t+\pi_s)$, from (3.2.1.) we obtain

$$\alpha_s(t+\pi_s) = \alpha_s\left(\frac{\pi_s}{2} - \left(-\frac{\pi_s}{2} - t\right)\right) = \beta_s\left(-\frac{\pi_s}{2} - t\right). \tag{3.2.8}$$

Suppose s is even, then the alpha function is odd and beta function is even, hence

$$\beta_s\left(-\frac{\pi_s}{2} - t\right) = \beta_s\left(\frac{\pi_s}{2} - (-t)\right) = \alpha_s(-t) = -\alpha_s(t). \tag{3.2.9}$$

We can apply the similar method to the beta function. Then we conclude

$$\begin{cases} \alpha_s(t+\pi_s) = -\alpha_s(t) \\ \beta_s(t+\pi_s) = -\beta_s(t) \end{cases}, \text{ when } s \text{ is even.} \tag{3.2.10}$$

If we now replace π_s by $t+\pi_s$ in (3.2.10) we get

$$\begin{cases} \alpha_s(t+2\pi_s) = \alpha_s(t) \\ \beta_s(t+2\pi_s) = \beta_s(t) \end{cases}, \text{ when } s \text{ is even.} \tag{3.2.11}$$

Therefore, alpha and beta function of even integer order are periodic functions with the period $2\pi_s$.

3.3. Minkowski Space

Suppose we have two independent variables x and y. We want to use the alpha and beta functions to generate a "polar coordinate". Let

$$\begin{cases} x = r_s\beta_s(\theta) \\ y = r_s\alpha_s(\theta) \end{cases}. \tag{3.3.1}$$

By (2.3.5), we obtain

$$r_s^s = x^s + y^s. \tag{3.3.2}$$

We want to show the transformation (3.1.1) cover the whole $\Re\times\Re$ space. We extend our understanding of trigonometric functions and say $\Re\times\Re \sim [0,2\pi_s)\times[0,\infty)$ by this transformation. Hence, by (3.1.2) we are in the s-Minkowski Space.

4. Generalized Laplace's Equation

Laplace's equation is well known by this application in physics. It is a standard problem to change the Laplace's equation to polar coordinate; this arises in connection with the Dirichlet problem, for example. Moreover, when we consider our polar coordinate in Minkowski space, a furthermore interesting result appears. In this section I will first reproduce the standard procedure to solving the Dirichlet Problem, namely, obtain the polar form of Laplace's equation. Then I will extend it by the similar method into the Minkowski space and present how to get the generalized Laplace's equation.

4.1. The Dirichlet Problem

Recall the Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 . \qquad (4.1.1)$$

We have $x = r\cos\theta$ and $y = r\sin\theta$. Our goal is to write (10) in terms of r and θ. The partial differential of x and y are

$$\begin{cases} \frac{\partial x}{\partial r} = \cos\theta \\ \frac{\partial y}{\partial r} = \sin\theta \end{cases}, \quad \begin{cases} \frac{\partial x}{\partial \theta} = -r\sin\theta \\ \frac{\partial y}{\partial \theta} = r\cos\theta \end{cases}. \qquad (4.1.2)$$

Therefore, we obtain

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \qquad (4.1.3)$$

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} \qquad (4.1.4)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y} \qquad (4.1.5)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r\left(\sin\theta\frac{\partial^2 u}{\partial x\partial\theta} + \cos\theta\frac{\partial u}{\partial x}\right) + r\left(\cos\theta\frac{\partial^2 u}{\partial y\partial\theta} - \sin\theta\frac{\partial u}{\partial y}\right)$$

$$= -r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial\theta}\right) - r\cos\theta\frac{\partial u}{\partial x} + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\theta}\right) - r\sin\theta\frac{\partial u}{\partial y}$$

$$= r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} - r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y}. \qquad (4.1.6)$$

Some terms in $\partial^2 u/\partial\theta^2$ appeared in $\partial^2 u/\partial r^2$ and $\partial u/\partial r$. In fact, if we look carefully, we can find that

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}. \qquad (4.1.7)$$

Hence, we get the polar form of Laplace's equation

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0. \qquad (4.1.8)$$

4.2. Dirichlet Problem in Minkowski Space and Generalized Laplace's Equation

We want to write the equation into the polar coordinate in Minkowski Space. As we discuss in section 3.3, let $x = r_s\beta_s(\theta)$ and $y = r_s\alpha_s(\theta)$. Then we have

$$\begin{cases}\dfrac{\partial x}{\partial r_s} = \beta_s \\ \dfrac{\partial y}{\partial r_s} = \alpha_s\end{cases}, \quad \begin{cases}\dfrac{\partial x}{\partial\theta} = -r_s\alpha_s^{s-1} \\ \dfrac{\partial y}{\partial\theta} = r_s\beta_s^{s-1}\end{cases} \qquad (4.2.1)$$

Therefore,

$$\frac{\partial u}{\partial r_s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r_s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r_s} = \beta_s\frac{\partial u}{\partial x} + \alpha_s\frac{\partial u}{\partial y} \qquad (4.2.2)$$

$$\frac{\partial^2 u}{\partial r_s^2} = \beta_s \frac{\partial^2 u}{\partial x \partial r_s} + \alpha_s \frac{\partial^2 u}{\partial y \partial r_s} = \beta_s \left(\frac{\partial^2 u}{\partial x \partial x} \frac{\partial x}{\partial r_s} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial r_s} \right) + \alpha_s \left(\frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial r_s} + \frac{\partial^2 u}{\partial y \partial y} \frac{\partial y}{\partial r_s} \right)$$

$$= \beta_s \left(\beta_s \frac{\partial^2 u}{\partial x^2} + \alpha_s \frac{\partial^2 u}{\partial x \partial y} \right) + \alpha_s \left(\beta_s \frac{\partial^2 u}{\partial y \partial x} + \alpha_s \frac{\partial^2 u}{\partial y^2} \right)$$

$$= \beta_s^2 \frac{\partial^2 u}{\partial x^2} + 2\alpha_s \beta_s \frac{\partial^2 u}{\partial x \partial y} + \alpha_s^2 \frac{\partial^2 u}{\partial y^2} \tag{4.2.3}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = -r_s \alpha_s^{s-1} \frac{\partial u}{\partial x} + r_s \beta_s^{s-1} \frac{\partial u}{\partial y} \tag{4.2.4}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -r_s \frac{\partial}{\partial \theta} \left(\alpha_s^{s-1} \frac{\partial u}{\partial x} \right) + r_s \frac{\partial}{\partial \theta} \left(\beta_s^{s-1} \frac{\partial u}{\partial y} \right)$$

$$= -r_s \left(\frac{\partial \alpha_s^{s-1}}{\partial \theta} \frac{\partial u}{\partial x} + \alpha_s^{s-1} \frac{\partial^2 u}{\partial x \partial \theta} \right) + r_s \left(\frac{\partial \beta_s^{s-1}}{\partial \theta} \frac{\partial u}{\partial y} + \beta_s^{s-1} \frac{\partial^2 u}{\partial y \partial \theta} \right)$$

$$= -r_s \left((s-1)\alpha_s^{s-2} \beta_s^{s-1} \frac{\partial u}{\partial x} + \alpha_s^{s-1} \left(\frac{\partial^2 u}{\partial x \partial x} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} \right) \right)$$
$$+ r_s \left((s-1)\beta_s^{s-2} \alpha_s^{s-1} \frac{\partial u}{\partial y} + \beta_s^{s-1} \left(\frac{\partial^2 u}{\partial y \partial x} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial y} \frac{\partial y}{\partial \theta} \right) \right)$$

$$= -r_s \left((s-1)\alpha_s^{s-2} \beta_s^{s-1} \frac{\partial u}{\partial x} + \alpha_s^{s-1} \left(-r_s \alpha_s^{s-1} \frac{\partial^2 u}{\partial x^2} + r_s \beta_s^{s-1} \frac{\partial^2 u}{\partial x \partial y} \right) \right)$$
$$+ r_s \left((s-1)\beta_s^{s-2} \alpha_s^{s-1} \frac{\partial u}{\partial y} + \beta_s^{s-1} \left(-r_s \alpha_s^{s-1} \frac{\partial^2 u}{\partial y \partial x} + r_s \beta_s^{s-1} \frac{\partial^2 u}{\partial y^2} \right) \right)$$

$$\begin{aligned} &= r_s^2 \alpha_s^{2s-2} \frac{\partial^2 u}{\partial x^2} - 2r_s^2 \alpha_s^{s-1} \beta_s^{s-1} \frac{\partial^2 u}{\partial x \partial y} + r_s^2 \beta_s^{2s-2} \frac{\partial^2 u}{\partial y^2} \\ &- r_s (s-1)\alpha_s^{s-2} \beta_s^{s-1} \frac{\partial u}{\partial x} - r_s (s-1)\beta_s^{s-2} \alpha_s^{s-1} \frac{\partial u}{\partial y} \end{aligned} \tag{4.2.5}$$

There are some terms in equation (5.2.5) appeared in (5.2.2) and (5.2.3). Although these equations are long, we can still simplifies them by writing down

$$\begin{aligned}&r_s^{s-4}\frac{\partial^2 u}{\partial\theta^2}+(s-1)r^{s-3}\alpha_s^{s-2}(\theta)\beta_s^{s-2}(\theta)\frac{\partial u}{\partial r_s}+r^{s-2}\alpha_s^{s-2}(\theta)\beta_s^{s-2}(\theta)\frac{\partial^2 u}{\partial r_s^2}\\&=y^{s-2}\frac{\partial^2 u}{\partial x^2}+x^{s-2}\frac{\partial^2 u}{\partial y^2}\end{aligned}. \qquad (4.2.6)$$

Set the above equation be zero, we have

$$y^{s-2}\frac{\partial^2 u}{\partial x^2}+x^{s-2}\frac{\partial^2 u}{\partial y^2}=0. \qquad (4.2.7)$$

Define $\nabla_s^2 = x^{2-s}\,\partial^2/\partial x^2 + y^{2-s}\,\partial^2/\partial y^2$ and suppose $x \neq 0$ and $y \neq 0$, (5.2.7) becomes

$$\nabla_s^2 u = x^{2-s}\frac{\partial^2 u}{\partial x^2}+y^{2-s}\frac{\partial^2 u}{\partial y^2}=0 \qquad (4.2.8)$$

Which is our generalized Laplace's Equation

4.3. The Solution of Generalized Laplace's Equation as a Power Series

As the standard way to solve partial differential equations, I use separating variable to obtain two ordinary differential equations. Let $u = X(x)Y(y)$

$$Y(y)x^{2-s}\frac{\partial^2 X(x)}{\partial x^2}+X(x)y^{2-s}\frac{\partial^2 Y(y)}{\partial y^2}=0 \qquad (4.3.1)$$

$u=0$ is a trivial solution. So we are now concentrate for the cases such that $u = X(x)Y(y) \neq 0$. Divide both sides by $X(x)Y(y)$, we obtain

$$\frac{x^{2-s}}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}+\frac{y^{2-s}}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=0 \qquad (4.3.2)$$

We argue that our solutions need to satisfy the equation although we vary x but keep y constant, or vary y but keep x constant. Therefore, each term need to be a constant. And these constants add up to be zero. Let an arbitrary constant k^s and we end up with two ordinary differential equations:

$$\begin{cases} \dfrac{d^2X(x)}{dx^2} = k^s x^{s-2} X(x) \\ \dfrac{d^2Y(x)}{dy^2} = -k^s y^{s-2} Y(y) \end{cases} \tag{4.3.3}$$

We first solve the “X” part of the separated equation. Let

$$X(x) = \sum_{n=0}^{\infty} c_n x^n \tag{4.3.4}$$

Put our power series form of $X(x)$ into the first equation of (5.3.3), we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} k^s c_n x^{n+s-2} \tag{4.3.5}$$

$$\sum_{n=0}^{s-3} (n+2)(n+1)c_{n+2} x^n + \sum_{n=s-2}^{\infty} (n+2)(n+1)c_{n+2} x^n = \sum_{n=s-2}^{\infty} k^s c_{n-s+2} x^n \tag{4.3.6}$$

Comparing the coefficients, we conclude that c_0 and c_1 are arbitrary constants, $c_2 = c_3 = ... = c_{s-1} = 0$. Moreover, for $n \geq 0$, we have

$$(n+2)(n+1)c_{n+2} = k^s c_{n-s+2} \tag{4.3.7}$$

$$c_{n+s} = \frac{k^s}{(n+s)(n+s-1)} c_n \tag{4.3.8}$$

In terms of c_0 and c_1, our series becomes

$$\begin{aligned} X(x) &= c_0 + c_1 x + \frac{k^s}{s(s-1)} c_0 x^s + \frac{k^s}{(s+1)s} c_1 x^{s+1} \\ &+ \frac{k^{2s}}{2s(2s-1)s(s-1)} c_0 x^{2s} + \frac{k^{2s}}{(2s+1)2s(s+1)s} c_0 x^{2s+1} + ... \end{aligned} \tag{4.3.9}$$

We group this into two one in terms of c_0 and the other into c_1, which is justified later,

$$X(x) = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{k^{ns}}{\prod_{m=1}^{n} ms(ms-1)} x^{ns} \right) + c_1 \left(x + \sum_{n=1}^{\infty} \frac{k^{ns}}{\prod_{m=1}^{n} (ms+1)ms} x^{ns+1} \right) \tag{4.3.10}$$

Is this series convergence? I use the ratio test to check both terms of c_0 and c_1.

$$\lim_{n\to\infty}\left|\frac{k^{(n+1)s}x^{(n+1)s}}{\prod_{m=1}^{n+1} ms(ms-1)}\frac{\prod_{m=1}^{n} ms(ms-1)}{k^{ns}x^{ns}}\right| = \left|x^s\right|\lim_{n\to\infty}\left|\frac{k^s}{(n+1)s((n+1)s-1)}\right| = 0 \qquad (4.3.11)$$

$$\lim_{n\to\infty}\left|\frac{k^{(n+1)s}x^{(n+1)s}}{\prod_{m=1}^{n+1}(ms+1)ms}\frac{\prod_{m=1}^{n}(ms+1)ms}{k^{ns}x^{ns}}\right| = \left|x^s\right|\lim_{n\to\infty}\left|\frac{k^s}{((n+1)s+1)(n+1)s}\right| = 0 \qquad (4.3.12)$$

Hence both of them are absolutely convergent for any x. This justifies our rearranging in (4.3.10). Using the similar method, we find the solution of "Y" part by the similar method

$$Y(y) = c'_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n k^{ns}}{\prod_{m=1}^{n} ms(ms-1)}y^{ns}\right) + c'_1\left(y+\sum_{n=1}^{\infty}\frac{(-1)^n k^{ns}}{\prod_{m=1}^{n}(ms+1)ms}y^{ns+1}\right) \qquad (4.3.13)$$

It is clear that the two terms on the right of (4.3.13) are convergence for any y.

The reason I used k^s to be our constant is we can group that with our independent variable. Namely, let $A = c_0$, $B = c_1/k$, equation (4.3.13) reduces to

$$X(x) = A\left(1+\sum_{n=1}^{\infty}\frac{(kx)^{ns}}{\prod_{m=1}^{n} ms(ms-1)}\right) + B\left(kx+\sum_{n=1}^{\infty}\frac{(kx)^{ns+1}}{\prod_{m=1}^{n}(ms+1)ms}\right). \qquad (4.3.14)$$

Let $C = c'_0$, $D = c'_1/k$, equation (4.3.13) reduces to

$$Y(y) = C\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(ky)^{ns}}{\prod_{m=1}^{n} ms(ms-1)}\right) + D\left(ky+\sum_{n=1}^{\infty}\frac{(-1)^n(ky)^{ns+1}}{\prod_{m=1}^{n}(ms+1)ms}\right) \qquad (4.3.15)$$

Consider the functions

$$a_s(x) = 1+\sum_{n=1}^{\infty}\frac{(kx)^{ns}}{\prod_{m=1}^{n} ms(ms-1)}. \qquad (4.3.16)$$

$$b_s(t) = t + \sum_{n=1}^{\infty} \frac{t^{ns+1}}{\prod_{m=1}^{n} (ms+1)ms}. \quad (4.3.17)$$

$$c_s(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{ns}}{\prod_{m=1}^{n} ms(ms-1)} \quad (4.3.18)$$

$$d_s(t) = t + \sum_{n=1}^{\infty} \frac{(-1)^n t^{ns+1}}{\prod_{m=1}^{n} (ms+1)ms} \quad (4.3.19)$$

Then (4.3.14) and (4.3.15) can be written as

$$\begin{cases} X(x) = Aa_s(kx) + Bb_s(kx) \\ Y(x) = Cc_s(ky) + Dd_s(ky) \end{cases} \quad (4.3.20)$$

Hence, a particular solution to the generalized Laplace's equation has the form

$$u_k(x, y) = (Aa_s(kx) + Bb_s(kx))(Cc_s(ky) + Dd_s(ky)) \quad (4.3.21)$$

Since the generalized Laplace's equation is linear, we can add up the solution for obtain a new solution. Therefore, the most general solution is

$$u(x, y) = \sum (Aa_s(kx) + Bb_s(kx))(Cc_s(ky) + Dd_s(ky)) \quad (4.3.22)$$

5. Reference

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